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SOME APPLICATIONS OF DIFFERENTIAL GEOMETRY TO THE INTERPRETATION OF PHYSICAL PHENOMENA*

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1. Introduction. The motivation of the present investigation was the conviction that a differential geometric analysis of such physical phenomena as heat conduction in a homogeneous medium and electrical charge distribution on closed conductors should yield interesting quantitative geometric interpretations. Although the study has been somewhat rudimentary, the results have been, in the author's opinion, of sufficient interest to warrant extending them to their mathematical analogues in n-dimensional Euclidean space. For the sake of elegance of presentation, the three-dimensional results will appear as specializations of the n-dimensional theory.

Let ϕ, ψ denote single-valued point functions of class C^2 in a domain D of a Euclidean space R_n of n-dimensions. Let V_{n-1} denote the hypersurface on which ϕ is constant which passes through a given point P of D. Let the family of such hypersurfaces which pass through points of D be denoted by F. Let σ denote arc length of the orthogonal trajectory T of F at P, the positive and negative senses of σ along T being taken to be those in which ϕ decreases and increases, respectively. The positive function Ω defined by the relation:

$$\Omega = -\partial\phi/\partial\sigma \tag{1.1}$$

will be called the "gauge function" of F at P. A locus of points on V_{n-1} at which Ω has a constant value is an $n-2$ dimensional variety V_{n-2} which will be called an "isogauge" of V_{n-1}. An isogauge of V_{n-1} with respect to F is, in general, an $n-2$ dimensional variety. It can be $n-1$ dimensional only if it coincides with V_{n-1}. In such a case, F is a family of concentric hyperspheres, each V_{n-1} being defined by a constant value of a differentiable function of the radius.

Let M denote the mean curvature of V_{n-1} at P, and let $\mathbf{p}$ and K_T denote the unit principal normal and curvature, respectively, of T at P. Let $\Delta\psi$, $\Delta_1\psi$ denote the Laplacians of ψ with respect to R_n and V_{n-1} respectively. The following theorems will be proved.

Theorem 1. *The relation which exists between the two Laplacians* $\Delta\psi$, $\Delta_1\psi$ *is given by*

$$\Delta\psi = \frac{\partial^2\psi}{\partial\sigma^2} - M\frac{\partial\psi}{\partial\sigma} + \Delta_1\psi - K_T\mathbf{p}\cdot\nabla\psi, \tag{1.2}$$

in which $\mathbf{p}\cdot\nabla\psi$ *is the projection of the gradient of* ψ *on the principal normal of* T *at* P.

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COROLLARY (1.1). *If ϕ is harmonic*

$$M = \frac{\partial}{\partial \sigma}(\ln \Omega), \tag{1.3}$$

in which $\Omega = -\partial\phi/\partial\sigma$, and M is the mean curvature of V_{n-1} at P (a result due to H. Weyl [5, p. 181]).

Theorem 2. Let τ denote the orthogonal trajectory on V_{n-1} at P of the family of isogauges of V_{n-1}, and let s denote its arc length. The principal normal **p** *of T at P is tangent to τ, and the curvature of T at P is given by the relation*

$$K_T = \frac{\partial}{\partial s}(\ln \Omega), \tag{1.4}$$

An application to steady-state heat flow may be described as follows. Let F be a family of isothermal surfaces with respect to heat flow in a homogeneous isotropic medium in ordinary space. The fundamental hypothesis in steady-state heat flow states that heat flows orthogonally across the isothermal surfaces at a rate proportional to $-\partial\phi/\partial\sigma$, in which ϕ denotes temperature. It follows that Corollary (1.1) and Theorem 2, in combination, yield the following interesting result.

Theorem 3. In the case of a steady-state heat flow in a homogeneous isotropic medium the ratio of the rates of flow at any two points P_0, P_1 of the medium is given by the relation

$$\Omega_0/\Omega_1 = \exp\left(-\int_{\sigma_0}^{\sigma} M\, d\sigma + \int_{s}^{s_1} K_T\, ds\right), \tag{1.5}$$

in which the first integral is along the arc of T from P_0 to the point P of intersection of T with the isothermal surface which passes through P_1 and the second is along the arc of τ from P to the intersection Q_1 of τ with the "isogauge" which passes through P_1.

It is known that on a closed charged conductor S_0 of class C^2 the density δ of charge at a point P_0 of S_0, if no other charges than that on the conductor are present, is proportional to $(-\partial\phi/\partial\sigma)_0$. From corollary (1.1) it follows that δ *is given by the relation*

$$\delta = k\Omega_1 \exp \int_{\sigma_0}^{\sigma_1} -M\, d\sigma, \tag{1.6}$$

in which Ω is the value of the gauge function at the point P_1 to which σ_1 corresponds, and the integration is from P_0 to P_1 along T.

It is known [4, p. 191] that electrical charge placed on an ellipsoidal conductor E_0 becomes distributed so that its density at a point P_0 of E_0 is proportional to the "supporting function" of E_0 at P_0, that is, the distance h_0 from the center of E_0 to the tangent plane to E_0 at P_0. The equipotential surfaces induced by the charged ellipsoid are confocal ellipsoids. Let E denote an equipotential ellipsoid whose potential is given by $\phi = c$. At points P_0, P of E_0, E, respectively, the gauge functions are given by

$$\Omega_0 = k_0 h_0, \qquad \Omega_c = k_c h, \tag{1.7}$$

in which k_0, k_c are constants, and Ω_0 is proportional to the density of charge of E_0 at P. On putting E_0, E in the roles of S_0, S for interpretation of (1.6) relative to confocal ellipsoids and making use of (1.7), the following relation is obtained.

$$mh_0/h = \exp\left(\int_\sigma^{\sigma_0} - M\, d\sigma\right), \tag{1.8}$$

in which $m = k_0/k_c$ is a constant which corresponds to the selection E_0 , E.

The simple yet fundamental role of the "supporting function" in characterizing physical phenomena associated with confocal ellipsoids suggests its usefulness in a general study of isothermal surfaces. Such a study is presented in the present paper in which use is made of the supporting function together with an associated "natural" coordinate system. The representation employed may be described as follows. Let a one-parameter family of surfaces of class C^2 be defined by a functional relation,

$$h = h(\theta, \omega, \phi), \qquad \phi = \text{constant}, \tag{1.9}$$

in which relative to a point $P(\theta, \omega, \phi)$ the numbers h, θ, ω, ϕ have the following meanings. One surface S on which ϕ is constant passes through P. The numbers h, θ, ω are spherical coordinates, with respect to some fixed rectangular cartesian coordinate frame, of the foot of the perpendicular line from the origin to the tangent plane to S at P; that is to say, h is the supporting function of S at P. A curve described by a point P as ϕ varies, but θ and ω are held fixed, is called a ϕ-curve. A curve is a ϕ-curve relative to F if, and only if, at its points the tangent planes to F are parallel.

Let a system F of surfaces, not necessarily isothermal, be represented by an equation of form (1.9). At a point P of a surface S of the system, let γ denote the acute angle between the normal to S at P and the tangent to the ϕ-curve at P, and let K denote the curvature at P of the curve of section of S by that normal plane of S which contains the tangent to the ϕ-curve at P. The following relations will be established

$$h_3 \frac{\partial h_3}{\partial \sigma} = h_{33} + (K \tan^2 \gamma) h_3^2 , \tag{1.10}$$

$$h_3^2\, \Delta\phi + M h_3 + \frac{\partial h_3}{\partial \sigma} = 0, \tag{1.11}$$

where

$$h_1 = \partial h/\partial\theta,\ h_2 = \partial h/\partial\omega,\ h_3 = \partial h/\partial\phi.$$

If ϕ is a harmonic function, relations (1.10), (1.11) reduce to

$$h_{33} + h_3^2(M + K \tan^2 \gamma) = 0, \qquad \frac{\partial h_3}{\partial \sigma} + M h_3 = 0. \tag{1.12}$$

The coefficients of Eqs. (1.12) *are geometric quantities whose values are independent of the choice of coordinates. The quantity* $-1/h_3$ will be shown to be *equal to the gauge function* Ω *of* F *at* P. It follows that the first equation of (1.12) is equivalent to the following first-order equation in ϕ and Ω

$$\frac{\partial \Omega}{\partial \phi} + M + K \tan^2 \gamma = 0. \tag{1.13}$$

This equation leads to the following integral equation which defines the difference between the gauge functions at two points P_0 , P, of a ϕ-curve

$$\Omega_1 - \Omega_0 = -\int_{\phi_0}^{\phi_1} (M + K \tan^2 \gamma)\, d\phi. \tag{1.14}$$

An interesting physical interpretation of (1.14) may be stated as follows.

Theorem 4. *In steady-state heat flow through a homogeneous isotropic medium, the difference between the rates of flow* ρ_0 , ρ_1 *at points* P_0 , P_1 , *respectively of a flow line is given by a relation of the form*

$$\rho_0 - \rho_1 = k \int_{\phi_0}^{\phi_1} (M + K \tan^2 \gamma)\, d\phi, \tag{1.15}$$

in which k is a positive constant and ϕ_0 , ϕ_1 *are temperatures at* P_0 , P_1 *respectively, and the integration is along the flow line.*

The results stated above may be specialized to obtain the analogous geometric interpretations of physical phenomena applicable to the systems of cylinders which constitute the integral surfaces of the Laplace equation in two variables.

2. The Laplacians Δ and Δ_1 of a function ψ with respect to the surfaces ϕ = constant. In a domain D of a Euclidean space R_n let ϕ, ψ denote single-valued point functions of class C^2, let the family F of hypersurfaces V_{n-1} (introduced in Sec. 1) be coordinate hypersurfaces $x^n = \phi = c$ whose orthogonal trajectories T are the x^n-curves of R_n . Let V_{n-1} be referred to an orthogonal system of coordinate curves. The fundamental metric then assumes the form[1]

$$ds^2 = g_{ii}(dx^i)^2 = g_{nn}(dx^n)^2 + g_{\alpha\alpha}(dx^\alpha)^2. \tag{2.1}$$

Theorem 1 will be proved and some specializations will be noted in this section.

The Laplacian of a function ψ with respect to R_n at P is given by the well-known formula

$$\Delta\psi = g^{ij}\left(\frac{\partial^2\psi}{\partial x^i\, \partial x^j} - \Gamma^h_{ij}\frac{\partial\psi}{\partial x^h}\right), \tag{2.2}$$

in which the functions g^{ij} are defined by relations of the form

$$g^{ij}g_{jk} = \delta^k_i = \begin{cases} 0, & k = i \\ 1, & k \neq i \end{cases} \tag{2.3}$$

and the functions Γ^h_{ij} are the Christoffel functions of the second kind defined by the relations

$$\Gamma^h_{ij} = g^{hk}[ij, k] = \frac{g^{hk}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{2.4}$$

In view of the conditions of orthogonality

$$g_{ij} = 0, \qquad i \neq j \tag{2.5}$$

and Eqs. (2.3) and (2.4), Eq. (2.2) assumes the form

$$\Delta\psi = g^{nn}\frac{\partial^2\psi}{\partial x^{n^2}} - g^{nn}\Gamma^n_{nn}\frac{\partial\psi}{\partial x^n} - g^{\alpha\alpha}\Gamma^n_{\alpha\alpha}\frac{\partial\psi}{\partial x^n} - g^{nn}\Gamma^\gamma_{nn}\frac{\partial\psi}{\partial x^\gamma} + \Delta_1\psi, \tag{2.6}$$

[1]Greek indices will have the range 1,2,...,n-1. Latin indices (except n) will have the range 1,2,...,n. Repeated indices in a term denote the usual summation convention over the respective ranges.

in which $\Delta_1\psi$ is defined by

$$\Delta_1\psi = g^{\alpha\beta}\left(\frac{\partial^2\psi}{\partial x^\alpha\,\partial x^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial\psi}{\partial x^\gamma}\right) \tag{2.7}$$

and the Christoffel functions are given by the formulas

$$\begin{aligned}
\Gamma^n_{nn} &= g^{nn}[nn, n] = \frac{g^{nn}}{2}\frac{\partial g_{nn}}{\partial x^n}, \\
\Gamma^n_{\alpha\beta} &= g^{nn}[\alpha\beta, n] = -\frac{g^{nn}}{2}\frac{\partial g_{\alpha\beta}}{\partial x^n}, \\
\Gamma^\gamma_{nn} &= g^{\gamma\omega}[nn, \omega] = -\frac{g^{\gamma\omega}}{2}\frac{\partial g_{nn}}{\partial x^\omega}.
\end{aligned} \tag{2.8}$$

The Laplacian of ψ may, therefore, be written in the form

$$\Delta\psi = g^{nn}\frac{\partial^2\psi}{(\partial x^n)^2} + \frac{g^{nn}g^{\alpha\alpha}}{2}\frac{\partial g_{\alpha\alpha}}{\partial x^n}\frac{\partial\psi}{\partial x^n} - g^{nn}\Gamma^i_{nn}\frac{\partial\psi}{\partial x^i} + \Delta\psi. \tag{2.9}$$

The following relations will be established

$$\frac{g^{nn}g^{\alpha\alpha}}{2}\frac{\partial g_{\alpha\alpha}}{\partial x^n}\frac{\partial\psi}{\partial x^n} = M\frac{\partial\psi}{\partial\sigma}, \tag{2.10}$$

$$\frac{\partial^2\psi}{\partial\sigma^2} - K_1 p^i\frac{\partial\psi}{\partial x^i} = g^{nn}\left(\frac{\partial^2\psi}{\partial x^{n^2}} - \Gamma^i_{nn}\frac{\partial\psi}{\partial x^i}\right), \tag{2.11}$$

in which p^i denotes the contravariant components of the unit principal normal of T at P.

Let ζ denote the unit normal to V_{n-1} at P, and let ζ^i denote its contravariant components. The following equations are well known [1, p. 32]

$$M = -\operatorname{div}\zeta = -\frac{1}{(g)^{1/2}}\frac{\partial}{\partial x^i}(\zeta^i g^{1/2}), \tag{2.12}$$

in which g denotes the determinant whose element in row i and column j is g_{ij}.

Equations (2.12) will be used to establish (2.10). Since ζ is the unit tangent to T at P, its contravariant components are defined by

$$\zeta^i = \frac{\partial x^i}{\partial x^n}\Big/\left(g_{ij}\frac{\partial x^i}{\partial x^n}\frac{\partial x^j}{\partial x^n}\right)^{1/2} = \delta^i_n / g^{1/2}_{nn}. \tag{2.13}$$

On substituting these values in (2.12) we find

$$-M = g^{-1/2}\frac{\partial}{\partial x^n}\left(\frac{g}{g_{nn}}\right)^{1/2} = \frac{1}{2g_{\alpha\alpha}(g_{nn})^{1/2}}\frac{\partial g_{\alpha\alpha}}{\partial x^n}. \tag{2.14}$$

Moreover, since $d\sigma^2 = g_{nn}(dx^n)^2$, we have

$$\frac{\partial\psi}{\partial\sigma} = \frac{\partial\psi}{\partial x^n}\frac{dx^n}{d\sigma} = g^{-1/2}_{nn}\frac{\partial\psi}{\partial x^n}. \tag{2.15}$$

Since $g^{ii} = 1/g_{ii}$ $(i = 1, 2, \cdots, n)$, Eq. (2.10) is an obvious consequence of (2.14) and (2.15).

The components $K_T p^i$ of the first curvature vector of T at P are defined by the intrinsic derivative of ζ^i along T. Thus, we have

$$K_T p^i = \zeta^i_{,j}\zeta^j, \qquad \zeta^i = \delta^i_n / g^{1/2}_{nn} . \tag{2.16}$$

Evaluation of the intrinsic derivative yields

$$\begin{aligned} K_T p^i &= \zeta^i_{,n} \Big/ g^{1/2}_{nn} = \left(\frac{\partial \zeta^i}{\partial x^n} + \zeta^h \Gamma^i_{hn}\right) \Big/ g^{1/2}_{nn} \\ &= -\frac{\delta^i_n}{2g^2_{nn}} \frac{\partial g_{nn}}{\partial x^n} + \frac{\Gamma^i_{nn}}{g_{nn}}. \end{aligned} \tag{2.17}$$

In view of (2.8) it follows from (2.17) that $K_T p^i \, \partial\psi/\partial x^i$ is given by

$$K_T p^i \frac{\partial \psi}{\partial x^i} = K_T p^\gamma \frac{\partial \psi}{\partial x^\gamma} = g^{nn}\Gamma^\gamma_{nn} \frac{\partial \psi}{\partial x^\gamma}. \tag{2.18}$$

Again making use of (2.8), simple calculations yield the following equations

$$\begin{aligned} \frac{\partial^2 \psi}{\partial \sigma^2} &= g^{nn} \frac{\partial^2 \psi}{(\partial x^n)^2} + \frac{1}{2} \frac{\partial g^{nn}}{\partial x^n} \frac{\partial \psi}{\partial x^n} \\ &= g^{nn} \frac{\partial^2 \psi}{(\partial x^n)^2} - \frac{(g^{nn})^2}{2} \frac{\partial g_{nn}}{\partial x^n} \frac{\partial \psi}{\partial x^n} = g^{nn}\left(\frac{\partial^2 \psi}{(\partial x^n)^2} - \Gamma^n_{nn} \frac{\partial \psi}{\partial x^n}\right). \end{aligned} \tag{2.19}$$

By subtracting the members of (2.18) from the corresponding members of (2.19), Eq. (2.11) is obtained. Finally, in view of Eqs. (2.10) and (2.11), which are now established, Eq. (2.9) may be written in the form of the relation which constitutes Theorem 1 in which $\mathbf{p}\cdot\nabla\psi$ denotes $p^i \, \partial\psi/\partial x^i$.

In the special case in which $\psi = \phi$, since $p^n = 0$ and x^n is independent of $x^1, x^2, \cdots, x^{n-1}$, the scalars $\Delta_1\phi$ and $p^i \, \partial\phi/\partial x^i$ vanish. Therefore we have

$$\Delta\phi = \frac{\partial^2 \phi}{\partial \sigma^2} - M \frac{\partial \phi}{\partial \sigma}. \tag{2.20}$$

Corollary 1.1 is an obvious consequence of (2.20).

In case the system of hypersurfaces $\phi = x^n = \text{constant}$ are parallel, x^n may be taken to be the arc σ measured from $x^n = 0$ along a normal. It follows that

$$g_{nn} = 1 = g^{nn}, \qquad \Gamma^i_{nn} = 0, \qquad (i = 1, 2, \cdots, n).$$

In this case the relation between the Laplacians of ϕ becomes

$$\Delta\phi - \Delta_1\phi = \frac{\partial^2 \phi}{\partial \sigma^2} - M \frac{\partial \phi}{\partial \sigma}. \tag{2.21}$$

In particular, for a V_1 in a surface, relative to a system of curves geodesically parallel [1, p. 57] to V_1, (2.21) reduces to

$$\Delta\phi = \frac{\partial^2 \phi}{\partial \sigma^2} - K \frac{\partial \phi}{\partial \sigma} + \frac{\partial^2 \phi}{\partial s^2}, \tag{2.22}$$

in which K is the curvature of V_1 at P and s is arc length of V_1 [3].

3. Proof of Theorem 2. On substituting from (2.8) into (2.17), the components of the first curvature vector of T at P are found to be given by

$$K_T p^\alpha = g^{\alpha\omega} \frac{\partial(\ln \Omega)}{\partial x^\omega}, \qquad p^n = 0, \tag{3.1}$$

in which $(g_{nn})^{-1/2} = \Omega = -\partial\phi/\partial\sigma$. On forming the inner product of the members of (3.1) with $g_{\alpha\beta}\, dx_\beta/ds$ in which dx_β/ds denotes the components p_β the following relation is obtained

$$K_T = \partial(\ln \Omega)/\partial s, \tag{3.2}$$

in which s denotes the arc length of the orthogonal trajectory on V_{n-1} at P of the family of isogauges of V_{n-1}. This completes the proof of Theorem 2.

4. The Laplacian $\Delta\phi$ in terms of the supporting function. Let h denote the supporting function of a one-parameter family of surfaces of class C^2 defined by a functional relation of form (1.9). The quantities θ, ω, ϕ defined in association with (1.9) will serve here as curvilinear coordinates of the point P of any surface. To express the Laplacian $\Delta\phi$ in terms of the partial derivatives of h with respect to θ, ω, ϕ, the components of the fundamental metric tensors g_{ij} and g^{ij} must first be calculated. For this purpose we make use of the definitions of h, θ, ω, ϕ in terms of a fixed auxiliary rectangular cartesian coordinate system (x, y, z). Since (θ, ω, ϕ) are curvilinear coordinates of the contact point P of a tangent plane to a ϕ = constant surface S, the rectangular cartesian coordinates (x, y, z) of P satisfy the following three equations. The first of these equations is the equation of the tangent plane at P, and the next two are obtained from it by partial differentiation with respect to θ and ω, respectively

$$\begin{aligned} x \sin\omega \cos\theta + y \sin\omega \sin\theta + z \cos\omega &= h, \\ -x \sin\omega \sin\theta + y \sin\omega \cos\theta &= h_1, \\ x \cos\omega \cos\theta + y \cos\omega \sin\theta - z \sin\omega &= h_2, \end{aligned} \tag{4.1}$$

in which $h_2 = \partial h/\partial\omega$, $h_1 = \partial h/\partial\theta$.

The following set of equations, which has the same solution as that of (4.1) and whose coefficients of x, y, z are functions solely of θ, is obtained by combining Eqs. (4.1) in the ways indicated by the forms of the right members.

$$\begin{aligned} x \cos\theta + y \sin\theta &= h \sin\omega + h_2 \cos\omega, \\ -x \sin\theta + y \cos\theta &= h_1 \csc\omega, \\ z &= h \cos\omega - h_2 \sin\omega. \end{aligned} \tag{4.2}$$

The solution of (4.2) is given by

$$\begin{aligned} x &= (h \sin\omega + h_2 \cos\omega) \cos\theta - h_1 \csc\omega \sin\theta, \\ y &= (h \sin\omega + h_2 \cos\omega) \sin\theta + h_1 \csc\omega \cos\theta, \\ z &= h \cos\omega - h_2 \sin\omega. \end{aligned} \tag{4.3}$$

On calculating the g's by making use of the relations

$$g_{ij} = \delta_{hk} \frac{\partial x^h}{\partial u^i} \frac{\partial x^k}{\partial u^j}, \qquad i, j = 1, 2, 3 \tag{4.4}$$

in which $x^1 = x, x^2 = y, x^3 = z, u^1 = \theta, u^2 = \omega, u^3 = \phi$, the following values are obtained

$$\begin{aligned} g_{11} &= e^2 \csc^2 \omega + f^2, & g_{12} &= f(g + e \csc^2 \omega), \\ g_{22} &= g^2 + f^2 \csc^2 \omega, & g_{13} &= -(fh_{23} + eh_{13} \csc^2 \omega), \\ g_{32} &= -(gh_{23} + fh_{13} \csc^2 \omega), & g_{33} &= h_3^2 + h_{23}^2 + h_{13}^2 \csc^2 \omega, \\ G &= | g_{ij} | = h_3^2 H^2 \csc^2 \omega, & H^2 &= (eg - f^2)^2, \end{aligned} \tag{4.5}$$

in which e, f, g are defined by the equations

$$\begin{aligned} e &= -(h \sin^2 \omega + h_2 \cos \omega \sin \omega + h_{11}), \\ f &= h_1 \cot \omega - h_{12}, \qquad g = -(h + h_{22}), \end{aligned}$$

the subscripts 1, 2, 3 of h denoting partial derivatives with respect to θ, ω, ϕ, respectively.

The functions e, f, g are actually the coefficients of the second fundamental form of S, as may be seen by direct calculations of these coefficients. Let $\mathbf{x}$, $\mathbf{x}_\alpha$, $\mathbf{x}_{\alpha\beta}$ denote the vectors whose ith components are x^i, $\partial x^i/\partial u^\alpha$, $\partial^2 x^i/\partial u^\alpha \partial u^\beta$ respectively. From the definition of the second fundamental form of S together with some orthogonality relations [6, pp. 92–94] it follows that the coefficients of the form are given by the set of determinant relations

$$d_{\alpha\beta} = (\mathbf{x}_{\alpha\beta} \mathbf{x}_1 \mathbf{x}_2)/| g_{\alpha\beta} |^{1/2}, \qquad \alpha, \beta = 1, 2 \tag{4.6}$$

in which the g's are defined in (4.5).

The following expressions for the components of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_{11}$, $\mathbf{x}_{12}$, $\mathbf{x}_{22}$ are obtained by differentiating Eqs. (4.3).

$$\begin{aligned} x_1^1 &= e \csc \omega \sin \theta - f \cos \omega \cos \theta, \\ x_2^1 &= f \csc \omega \sin \theta - g \cos \omega \cos \theta, \\ x_1^2 &= -e \csc \omega \cos \theta - f \cos \omega \sin \theta, \\ x_2^2 &= -g \cos \omega \sin \theta - f \csc \omega \cos \theta, \\ x_1^3 &= f \sin \omega, \\ x_2^3 &= g \sin \omega, \\ x_{11}^1 &= (e \csc \omega - f_1 \cos \omega) \cos \theta + (e_1 \csc \omega + f \cos \omega) \sin \theta, \\ x_{11}^2 &= (-e_1 \csc \omega - f \cos \omega) \cos \theta + (e \csc \omega - f_1 \cos \omega) \sin \theta, \\ x_{11}^3 &= f_1 \sin \omega, \\ x_{21}^1 &= (f \csc \omega - g_1 \cos \omega) \cos \theta + (f_1 \csc \omega + g \cos \omega) \sin \theta, \\ x_{21}^2 &= (-g \cos \omega - f_1 \csc \omega) \cos \theta + (f \csc \omega - g_1 \cos \omega) \sin \theta, \\ x_{21}^3 &= g_1 \sin \omega, \\ x_{22}^1 &= (g \sin \omega - g_2 \cos \omega) \cos \theta + (f_2 \csc \omega - f \csc \omega \cot \omega) \sin \theta, \\ x_{22}^2 &= (f \csc \omega \cot \omega - f_2 \csc \omega) \cos \theta + (g \sin \omega - g_2 \cos \omega) \sin \theta, \\ x_{22}^3 &= g_2 \sin \omega + g \cos \omega. \end{aligned} \tag{4.7}$$

The three-rowed determinants which appear in (4.6) may now be written in the form

$$(\mathbf{x}_{\alpha\beta}\mathbf{x}_1\mathbf{x}_2) = \begin{vmatrix} x^1_{\alpha\beta} & e\csc\omega\sin\theta - f\cos\omega\cos\theta & f\csc\omega\sin\theta - g\cos\omega\cos\theta \\ x^2_{\alpha\beta} & -e\csc\omega\cos\theta - f\cos\omega\sin\theta & -g\cos\omega\sin\theta - f\csc\omega\cos\theta \\ x^3_{\alpha\beta} & f\sin\omega & g\sin\omega \end{vmatrix}, \quad (4.8)$$

in which $x^i_{\alpha\beta}$ $(i = 1, 2, 3)$ are given by (4.7).

After transforming this determinant by elementary transformations on the rows (the transformations being indicated by the transformed elements of the first column), the following form is assumed

$$(\mathbf{x}_{\alpha\beta}\mathbf{x}_1\mathbf{x}_2) \quad (4.9)$$
$$= \begin{vmatrix} x^1_{\alpha\beta}+x^3_{\alpha\beta}\cot\omega\cos\theta+(x^2_{\alpha\beta}+x^3_{\alpha\beta}\cot\omega\sin\theta)\tan\theta & 0 & 0 \\ x^2_{\alpha\beta}+x^3_{\alpha\beta}\cot\omega\sin\theta & -e\csc\omega\cos\theta & -f\csc\omega\cos\theta \\ x^3_{\alpha\beta} & f\sin\omega & g\sin\omega \end{vmatrix}.$$

In view of Eqs. (4.5), (4.7), and (4.9), simple calculations yield the values e, f, g for the coefficients d_{11} , d_{12} , d_{22} of the second fundamental form, respectively.

The components of the contravariant metric tensor, calculated according to the definition

$$g^{ij} = (\text{cofactor of } g_{ij})/G,$$

are found to have the following values

$$\begin{aligned} g^{11} &= [f^2 + g^2\sin^2\omega + (gh_{13} - fh_{23})^2/h_3^2]/H^2, \\ g^{12} &= [(eh_{23} - fh_{13})(gh_{13} - fh_{23})/h_3^2 - f(e + g\sin^2\omega)]/H^2, \\ g^{13} &= (gh_{13} - fh_{23})/Hh_3^2\,, \qquad g^{22} = [e^2 + f^2\sin^2\omega + (fh_{13} - eh_{23})^2/h_3^2]/H^2, \\ g^{23} &= (eh_{23} - fh_{13})/Hh_3^2\,, \qquad g^{33} = 1/h_3^2\,. \end{aligned} \quad (4.10)$$

Since the absolute value of $\partial\phi/\partial\sigma$ is equal to the length of the gradient of ϕ, where the components of the gradient of ϕ are δ^3_i , we have the relations

$$\left(\frac{\partial\phi}{\partial\sigma}\right)^2 = (\nabla\phi)^2 = g^{ij}\delta^3_i\delta^3_j = g^{33} = 1/h_3^2\,. \quad (4.11)$$

For a system of closed surfaces ϕ = constant, according to the convention adopted for the sign of σ, the direction of increasing σ is outward. Therefore, if the origin is an interior point of the system, h increases as σ increases, and it follows that

$$\partial\phi/\partial\sigma = 1/h_3 = -\Omega. \quad (4.12)$$

The components η^i of the unit normal to the surface $S(\phi = c)$ at P are given by

$$\eta^i = h_3 g^{i3}. \quad (4.13)$$

From (4.12) and (4.13) it follows that

$$\frac{\partial^2\phi}{\partial\sigma^2} = -\frac{1}{h_3^2}\frac{\partial h_3}{\partial\sigma} = -\frac{1}{h_3}\left(\frac{\partial h_3}{\partial u^i}\right)(g^{i3}), \quad (4.14)$$

in which $u^1 = \theta$, $u^2 = \omega$, $u^3 = \phi$. On substituting the values of the g's as given by (4.10) into (4.14) and combining the result with (4.12) to evaluate the expression for $\Delta\phi$ given by (2.20) one readily obtains the relation

$$-h_3^2\,\Delta\phi = Mh_3^2 + h_{33} + (eh_{23}^2 - 2fh_{13}h_{23} + gh_{13}^2)/H. \tag{4.15}$$

In view of (4.14), Eq. (2.20) may now be written in the form

$$h_3^2\,\Delta\phi + Mh_3 + \frac{\partial h_3}{\partial\sigma} = 0 \tag{4.16}$$

which appears in Sec. 1 as (1.11).

The derivation of (1.10) proceeds as follows. Consider a point P' on the orthogonal trajectory T of S at P whose coordinates are $(\theta + d\theta, \omega + d\omega, \phi + d\phi)$. As P' varies along T tending toward P the following limiting relations are approached

$$d\omega/d\theta = g^{23}/g^{13}, \qquad d\phi/d\omega = g^{33}/g^{23}, \tag{4.17}$$

since the contravariant components of the gradient of ϕ are given by g^{i3}. Let Q denote the point of intersection of S with the ϕ-curve passing through P'. The coordinates of Q are $(\theta + d\theta, \omega + d\omega, \phi)$. It is now clear that the limit of the direction of QP as P' and Q tend to P is given by the first equation of (4.17). It is clear from the definitions of P' and Q relative to P that the limit of the direction of QP is the direction at P of the line of intersection of the tangent plane to S at P with the normal plane to S which contains the tangent to the ϕ-curve at P. The normal curvature of S at P which corresponds to this direction will be denoted by K. It follows that K is given by the relation

$$K = (e(g^{13})^2 + 2fg^{13}g^{23} + g(g^{23})^2)/g_{\alpha\beta}g^{\alpha 3}g^{\beta 3}. \tag{4.18}$$

On substituting from (4.4) into (4.12) we find

$$K = (eh_{23}^2 - 2fh_{13}h_{23} + gh_{13}^2)/H[h_{23}^2 + h_{13}^2\csc^2\omega]. \tag{4.19}$$

Let γ denote the acute angle between the normal to S at P and the tangent to the ϕ-curve at P. The covariant components ζ_i of the unit normal are given by

$$\zeta_i = \delta_i^3/(g^{33})^{1/2}. \tag{4.20}$$

The contravariant components ξ^i of the unit tangent to the ϕ-curve at P are defined by

$$\xi^i = \delta_3^i/(g_{33})^{1/2}. \tag{4.21}$$

It follows that $\cos\gamma$ is given by the relation

$$\cos\gamma = \zeta_i\xi^i = 1/(g^{33}g_{33})^{1/2} = -h_3/(h_3^2 + h_{23}^2 + h_{13}^2\csc^2\omega)^{1/2}. \tag{4.22}$$

Therefore $\tan^2\gamma$ is given by the relation

$$\tan^2\gamma = (h_{23}^2 + h_{13}^2\csc^2\omega)/h_3^2\,. \tag{4.23}$$

In view of this and (4.19), relation (4.15) may be written in the form

$$-h_3^3\,\Delta\phi = Mh_3^2 + h_{33} + h_3^2K\tan^2\gamma \tag{4.24}$$

which, in view of (4.16), is equivalent to (1.10).

If ϕ is a harmonic function, Eqs. (4.16) and (4.24) assume the forms

$$\frac{\partial h_3}{\partial \sigma} + Mh_3 = 0, \tag{4.25}$$

$$h_{33} + (M + K \tan^2 \gamma)h_3^2 = 0, \tag{4.26}$$

respectively.

In view of (4.12), Eq. (4.26) can be written in the form

$$\frac{\partial \Omega}{\partial \phi} + M + K \tan^2 \gamma = 0. \tag{4.27}$$

Equation (1.14) and Theorem 4 (stated in Sec. 1) now follow immediately.

5. The equation of Laplace in two variables. Consider a family F of $\phi =$ constant cylinders whose generators are parallel to a fixed axis, say the z-axis. The supporting function of a contact point P of a tangent plane to the $\phi =$ constant cylinder is independent of the position of P on the generator. The only admissible value of ω is, therefore, $\pi/2$ and the number triple $(\theta, \pi/2, \phi)$ determines a generator rather than a point of the generator. In order to apply results of Sec. 4 to this case it will be convenient to determine a point P of a generator by specifying its z coordinate. Thus we augment the original curvilinear coordinates θ, ϕ by the coordinate $u^2 = z$.

The auxiliary rectangular coordinates of P satisfy the set of equations

$$\begin{aligned} x \cos \theta + y \sin \theta &= h, \\ -x \sin \theta + y \cos \theta &= h_1 , \end{aligned} \tag{5.1}$$

in which $h = h(\theta, \phi)$, $h_1 = \partial h/\partial \theta$, $h_2 = \partial h/\partial z$, $h_3 = \partial h/\partial \phi$. The solution (x, y) of (5.1) is given by

$$\begin{aligned} x &= h \cos \theta - h_1 \sin \theta, \\ y &= h \sin \theta + h_1 \cos \theta. \end{aligned} \tag{5.2}$$

The g's of the metric tensor in terms of the coordinates $u^1 = \theta$, $u^2 = z$, $u^3 = \phi$, may be easily calculated. For this purpose, the following values of the partial derivatives are needed.

$$\begin{aligned} &\frac{\partial x}{\partial u^1} = -(h + h_{11}) \sin \theta, \quad \frac{\partial y}{\partial u^1} = (h + h_{11}) \cos \theta, \quad \frac{\partial z}{\partial u^1} = \theta, \\ &\frac{\partial x}{\partial u^2} = 0, \quad \frac{\partial y}{\partial u^2} = 0, \quad \frac{\partial z}{\partial u^2} = 1, \\ &\frac{\partial x}{\partial u^3} = h_3 \cos \theta - h_{13} \sin \theta, \quad \frac{\partial y}{\partial u^3} = h_3 \sin \theta + h_{13} \cos \theta, \quad \frac{\partial z}{\partial u^3} = 0. \end{aligned} \tag{5.3}$$

Since the g's are defined in terms of these partial derivatives by a set of equations of form (4.2) in which $x^1 = x$, $x^2 = y$, $x^3 = z$, the following values are readily obtained

$$\begin{aligned} g_{13} &= (h + h_{11})h_{13} , \quad g_{23} = 0, \quad g_{33} = h_3^2 + h_{13}^2 , \\ g_{11} &= (h + h_{11})^2, \quad g_{12} = 0, \quad g_{22} = 1. \end{aligned} \tag{5.4}$$

The quantity $(h + h_{11})^2$ will be shown to be the square of the radius of curvature of the curve C of normal section of a $\phi =$ constant cylinder. For this purpose, let α

denote the angle of inclination of the tangent to C at P with respect to the x-axis. Then $\alpha = \theta + \pi/2$. From the definition of curvature K of C, we have

$$K = d\alpha/ds = d\theta/ds,$$

where differentiation is with respect to arc length of C at P. On differentiating (5.2) with respect to s we find

$$\frac{dx}{ds} = -(h + h_{11}) \sin \theta \frac{d\theta}{ds}, \qquad \frac{dy}{ds} = (h + h_{11}) \cos \theta \frac{d\theta}{ds}. \tag{5.5}$$

Squaring (5.5), and adding yields

$$1 = (h + h_{11})^2 \left(\frac{d\theta}{ds}\right)^2 = K^2(h + h_{11})^2.$$

This is what was to be shown, since r^2 is by definition $1/K^2$.

It will be shown that the necessary and sufficient condition that ϕ be a solution of the equation of Laplace is that the function h be a solution of the equation

$$r \frac{\partial p}{\partial \phi} = p^2 + q^2, \tag{5.6}$$

in which r is the radius of curvature of C at P, and p and q are defined by the relations

$$p = \frac{\partial h}{\partial \phi}, \qquad q = \frac{\partial p}{\partial \theta}.$$

If the origin of rectangular coordinates is an interior point of the system of $\phi =$ constant cylinders, Eq. (4.12) holds. Recalling that the positive sense of σ is that in which ϕ decreases, and that the gauge function diminishes as P moves along a flow line (σ increasing) we find

$$\frac{\partial \Omega}{\partial \sigma} = -\frac{\partial^2 \phi}{\partial \sigma^2} = -\frac{\partial}{\partial \sigma}(h_3^{-1}) < 0, \qquad \Omega > 0. \tag{5.7}$$

A relation of form (4.14) holds for a system of $\phi =$ constant surfaces, independently of the choice of the coordinates u^1, u^2, and consequently, in the present coordinates $u^1 = \theta$, $u^2 = z$. Thus

$$-h_3^2 \frac{\partial^2 \phi}{\partial \sigma^2} = \frac{\partial h_3}{\partial \sigma} = h_3(h_{31}g^{13} + h_{33}g^{33}). \tag{5.8}$$

But in view of (4.16), $\partial h_3/\partial \sigma$ may be replaced by $-Mh_3$ if and only if ϕ is harmonic. On making this replacement and dividing by h_3, Eq. (5.8) assumes the form

$$-h_3 \frac{\partial^2 \phi}{\partial \sigma^2} = -M = h_{31}g^{13} + h_{33}g^{33}. \tag{5.9}$$

From (5.7), M is found to be negative. The mean curvature M which is the sum of the principal curvatures of the cylinder $\phi =$ constant at P, is given by the relation

$$M = -1/r, \tag{5.10}$$

since one of the principal curvatures is the curvature of C at P and the other is zero, the curvature of the generator at P.

The contravariant components g^{ij} are defined by the relations

$$g^{ij} = (\text{cofactor of } g_{ij})/|\, g_{ij}\,|, \qquad i, j = 1, 2, 3.$$

In view of (5.4) the determinant of the g's is given by the equation

$$g_{ij} = \begin{vmatrix} r^2 & 0 & rh_{13} \\ 0 & 1 & 0 \\ rh_{13} & 0 & h_3^2 + h_{13}^2 \end{vmatrix}.$$

It follows that g^{13}, g^{33} are given by the relations

$$g^{13} = -h_{13}/rh_3^2\,, \qquad g^{33} = 1/h_3^2\,. \tag{5.11}$$

On substituting from (5.10) and (5.11) into (5.9) and clearing, the resulting equation is

$$rh_{33} = (h_3^2 + h_{13}^2), \tag{5.12}$$

which is (5.6), since $h_3 = \partial h/\partial\phi = p$, $h_{13} = \partial p/\partial\theta$, $h_{33} = \partial p/\partial\phi$.

The theorems stated in the introduction can be easily specialized for the two dimensional case. The statements of these two dimensional theorems will be omitted here.

Additional results which are not specializations of the theorems hereto proved, but whose proofs involve only routine calculations will be presented here (without proof).

Theorem 5.1. *The ratio of the gauge functions at points P_0 , P of a ϕ-curve is given by the integral*

$$\Omega_0/\Omega = \exp \int_{s_0}^{s} (K^2 + K_T^2)^{1/2}\, ds \tag{5.13}$$

in which the integration is with respect to arc length of the ϕ-curve from P_0 to P, and K, K_T are the curvatures at P of the ϕ = constant curve (isothermal curve) at P and the orthogonal trajectory of the ϕ-curves at P, respectively.

Theorem 5.2. *If the density of charge at each point P of the curve C_0 of a clossd conductor is proportional to a constant power of the supporting function h of P, then the exponent of the power is unity and the curve C_0 is an ellipse, and the equipotential curves induced by the distribution of charge on C_0 are ellipses confocal with C_0 .*

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BOOK REVIEWS

Näherungsmethoden der höheren Analysis. By L. W. Kantorowitsch and W. I. Krylow. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. xi + 611 pp. $11.25.

This book is a German translation from the third Russian edition (1950), the latter being only a minor revision of the second Russian edition (1941). It presents a wealth of methods for the approximate solution of partial differential equations, the main interest being centered about elliptic equations, especially Laplace's and Poisson's equations, and the biharmonic equation, with the usual boundary conditions. In Chapter I methods based on the representation of the solution as an infinite series are treated with emphasis on Fourier series and orthogonal functions. A section of this chapter is devoted to infinite systems of linear equations. Chapter II contains methods for the approximate solution of the Fredholm integral equation together with an application to the problems of Dirichlet and Neumann. In Chapter III the differential equation is approximated by a corresponding difference equation and methods for solving the latter are studied. Chapter IV is one of the most interesting of the book and is devoted to variational methods. There is given a clear account of the methods of Ritz and Galerkin; in contrast to many other presentations of this subject the question of convergence is carefully examined and estimations of error are given for some simple but important cases. The remainder of the book, Chapters V-VII, deals with various questions concerning conformal mapping, application of conformal mapping to boundary problems for harmonic functions, and the alternating method of Schwarz.

The book presents an excellent survey of different analytical tools of importance in numerical analysis, the level being somewhat more advanced than is ordinarily the case in text-books on this subject. Theorems are proved in a stringent way and much emphasis is laid upon estimations of error. Several theorems can, however, be shown to be true under more general conditions than are stated here, but the generality of the assumptions is entirely sufficient for the applied mathematician. Every new method is illustrated by one or two well-chosen problems, which are worked out in detail. There are now exercises for the reader.

Since different methods mentioned in the book can often be applied to the same problem, it would have been valuable if the advantages and disadvantages of the respective methods had been discussed. As is usual in Russian mathematical text-books references are made almost exclusively to Russian papers. Since the book except for minor changes is the same as the 1941 edition, the development of numerical analysis during the last fifteen years has not influenced the contents, for instance the unified treatment of many methods and theorems which can be made by means of functional analysis is not used and the possibility of applying a computer is not discussed. These minor objections are, however, in no sense important. As a whole, the book is a very interesting and useful contribution to the literature on numerical analysis.

C. G. Esseen

Experimental designs. By William G. Cochran and Gertrude M. Cox. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1957. xiv + 611 pp. $10.25.

This book is "intended to serve as a handbook which is consulted when a new experiment is under consideration." The first edition was highly successful in this aim and the second edition is an augmentation of the first edition to take care of advances in the design of comparative experiments in the last few years.

This is the best book in existence on the down-to-earth practice of experimental design. There are aspects of the second edition which are unsatisfactory and these arise because the material of the first edition was transferred to the second edition *in toto*. Some of the material of the first edition could have been improved and the continuity of old and new material improved.

The book will be of very high value to experimenters and experimental statisticians and one can also surmise that workers in mathematical statistics will obtain some benefit in knowledge and outlook by perusing the book, since statistics is a subject which has in the past gained so much from understanding the needs of research workers.

O. Kempthorne

(*Continued on p. 396*)

MULTIPLE FOURIER ANALYSIS IN RECTIFIER PROBLEMS*

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Abstract. The non-linear problem of the multiple Fourier analysis of the output from a cut-off power law rectifier responding to a several-frequency input is reviewed for the one- and two-frequency problems and is briefly investigated for the three-frequency problem. The solutions for the modulation product amplitudes or multiple Fourier coefficients are obtained in exact although transcendental form. An account of the mathematical properties of these multiple Fourier coefficients or Bennett functions, including hypergeometric representations and power series expansions for them as well as recurrence relations satisfied by them, is given in the paper together with line graphs of the first ten basic functions for the one-frequency problem and of the first fifteen basic functions for the two-frequency problem. Further applications of the theory are also given to the computation of average output power with the aid of the multiple Fourier coefficients or Bennett functions studied in the paper, and the work is concluded with some brief remarks concerning the interpretation of the results in terms of the theory of almost periodic functions and the generalized Fourier series of Bohr under appropriate conditions. Numerical tables of the functions graphed have been prepared and are available separately in the United States and Great Britain for applications requiring great accuracy. Finally, the entire theory is based on the original method of the expansion of the rectifier output in multiple Fourier series introduced by Bennett in 1933 and 1947.

1. Introduction. A non-linear problem of continuing interest in theoretical electronics, and one much studied in the one- and two-frequency instances, is the multiple Fourier analysis of the output from a cut-off power law rectifier responding to a several-frequency input under various assumptions about the cut-off bias, about the exponent of the power law, and about the input amplitude ratios. Among others Bennett [1, 2], Kaufman [3, 4], Lampard [5], Salzberg [6], and Sternberg, Kaufman, Shipman, and Thurston [7, 8, 9, 10, 11] have obtained exact although transcendental solutions for the modulation product amplitudes or multiple Fourier coefficients in the first two cases mentioned and have tabulated some of these quantities under the name of Bennett functions, while Feuerstein [12], following a method of Bennett [1] and Rice [13], has recently given a quite extensive treatment of the n-frequency problem in terms of generalized Weber-Schafheitlin integrals [14] suitable for numerical evaluation.

In this paper, after reviewing some of the more interesting results noted above for the one- and two-frequency problems associated with a cut-off power law rectifier, with one or two new results added, we briefly investigate the three-frequency problem by a transcendental method of extension of previous results using, particularly, certain integral recurrence relations which hold between the multiple Fourier coefficients or Bennett functions of different multiplicities. Since the frequencies and phase angles are readily found, the real problem with which we are concerned in each instance consists of evaluating the single, double, or triple integrals defining the Fourier coefficients in the multiple Fourier series expansion of the output from the rectifier. The mathematical properties of these multiple Fourier coefficients or Bennett functions are investigated

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systematically, and hypergeometric representations and power series expansions for them and recurrence relations satisfied by them are given in the paper, together with line graphs of the first ten basic functions for the one-frequency problem and of the first fifteen basic functions for the two-frequency problem. In addition to providing for the multiple Fourier series expansion of the output from the rectifier, it is shown that the average output power may also be computed in terms of the same multiple Fourier coefficients or Bennett functions as those studied, provided, in the two- and three-frequency cases, that the input is non-periodic. No attempt is made to discuss modulation product problems more general than those associated with rectifiers, the reader in such instances being referred to Sternberg and Kaufman [7, 8, 9] for a general approximation process applicable to any continuous modulator, the solution again being formulated in terms of Bennett functions as here. The paper is concluded with some brief remarks concerning the interpretation of the results in terms of the theory of almost periodic functions and the generalized Fourier series of Bohr, again under the proviso, in the two- and three-frequency cases, that the input be non-periodic. Numerical tables giving values of the functions graphed good to 1×10^{-6} units at values of the arguments spaced one-tenth of a unit apart have been prepared and are available separately from the sources noted below in the United States and Great Britain for applications requiring great accuracy. Finally, we may note in passing that the entire theory is based on the original method of the expansion of the rectifier output in multiple Fourier series introduced by Bennett [1, 2] in 1933 and 1947.

The general rectifier problem is formulated precisely and the solution in simple or multiple Fourier series is outlined in Sec. 2. The review of results for the one- and two-frequency problems is presented in Secs. 3 and 4 while the three-frequency problem is treated in Sec. 5. The results of the theory which apply to the computation of average output power for a single frequency periodic input or a multiple frequency non-periodic input are given in Sec. 6, and the concluding remarks bearing on the connections with the theory of almost periodic functions and the generalized Fourier series of Bohr are given in Sec. 7. Copies of the numerical tables of the Bennett functions graphed in the paper and available to the public have been deposited with Mr. D. H. Lehmer of the Unpublished Mathematical Tables File in the United States and with Mr. Andrew Young of the Grace Library of the University of Liverpool in Great Britain and with the authors.

2. Formulation of the problem. Consider a cut-off power law rectifier having an output versus input characteristic $Y = Y^{\nu}(X; X_0)$ of the form

$$Y^{\nu}(X; X_0) = \begin{cases} (X - X_0)^{\nu}, & X > X_0, \\ 0, & X \leq X_0, \end{cases} \qquad \nu \geq 0. \tag{2.1}$$

Let the input to the rectifier be a one-, two-, or three-frequency function of time, $x(t)$, of one of the forms

$$x(t) = P \cos (pt + \theta_p), \qquad P > 0, \tag{2.2}$$

$$x(t) = P \cos (pt + \theta_p) + Q \cos (qt + \theta_q), \qquad P \geq Q > 0, \tag{2.3}$$

or

$$x(t) = P \cos (pt + \theta_p) + Q \cos (qt + \theta_q) + R \cos (rt + \theta_r), \qquad P \geq Q \geq R > 0. \tag{2.4}$$

The output from the rectifier, $y(t) \equiv Y^r[x(t); X_0]$ has then a single, double, or triple Fourier series expansion of the corresponding form

$$y(t) = \tfrac{1}{2}P^r A_0^{(r)}(h) + P^r \sum_{m=1}^{\infty} A_m^{(r)}(h) \cos(\omega_m t + \phi_m), \tag{2.5}$$

$$y(t) = \tfrac{1}{2}P^r A_{00}^{(r)}(h, k) + P^r \sum_{m,n=0}^{\infty *} A_{*mn}^{(r)}(h, k) \cos(\omega_{*mn} t + \phi_{*mn}), \tag{2.6}$$

or

$$y(t) = \tfrac{1}{2}P^r A_{000}^{(r)}(h, k_1, k_2) + P^r \sum_{m,n,l=0}^{\infty *} A_{**mnl}^{(r)}(h, k_1, k_2) \cos(\omega_{**mnl} t + \phi_{**mnl}), \tag{2.7}$$

where $h = X_0/P$, $k = Q/P > 0$, and $k_1 = Q/P \geq k_2 = R/P > 0$, and where the modulation product angular frequencies, phase angles, and amplitudes are given by the formulas

$$\omega_m = mp, \qquad \phi_m = m\theta_p, \tag{2.8}$$

$$\omega_{*mn} = mp \pm nq, \qquad \phi_{*mn} = m\theta_p \pm n\theta_q, \tag{2.9}$$

$$\omega_{**mnl} = mp \pm nq \pm lr, \qquad \phi_{**mnl} = m\theta_p \pm n\theta_q \pm l\theta_r, \tag{2.10}$$

and the formulas

$$A_m^{(r)}(h) = \frac{2}{\pi} \int_{\mathfrak{R}_1} (\cos u - h)^r \cos mu \, du, \tag{2.11}$$

$$\mathfrak{R}_1 : \cos u \geq h, \qquad 0 \leq u \leq \pi,$$

$$A_{*mn}^{(r)}(h, k) = \frac{2}{\pi^2} \iint_{\mathfrak{R}_2} (\cos u + k \cos v - h)^r \cos mu \, du \cos nv \, dv, \tag{2.12}$$

$$\mathfrak{R}_2 : \cos u + k \cos v \geq h, \qquad 0 \leq u, v \leq \pi,$$

$$A_{**mnl}^{(r)}(h, k_1, k_2) = \frac{2}{\pi^3} \iiint_{\mathfrak{R}_3} (\cos u + k_1 \cos v + k_2 \cos w - h)^r \cdot \cos mu \, du \cos nv \, dv \cos lw \, dw, \tag{2.13}$$

$$\mathfrak{R}_3 : \cos u + k_1 \cos v + k_2 \cos w \geq h, \qquad 0 \leq u, v, w \leq \pi,$$

where in all of these formulas the indices m, n, l take all integral values $m, n, l \geq 0$ and, finally, where the asterisks on the summation signs in (2.6) and (2.7) indicate that in these multiple Fourier series we sum only on all distinct arrangements of plus and minus signs, equivalent arrangements being taken only with the plus signs and with the zero order terms, particularly, having been removed from the sums.

We will assume the genesis of these Fourier expansions to be well-known and will refer the reader to Bennett [1] or Sternberg and Kaufman [7] for additional details of such matters. It may be worth noting, however, that the existence of these Fourier series follows at once from the continuity of the kernel functions in the integrands of the quantities $A_m^{(r)}(h)$, $A_{*mn}^{(r)}(h, k)$, and $A_{**mnl}^{(r)}(h, k_1, k_2)$ within the corresponding regions $\mathfrak{R}_1$, $\mathfrak{R}_2$, and $\mathfrak{R}_3$ while the convergence of these Fourier series to the functions $y(t)$ follows

at least at all points of continuity of the $y(t)$, in the case of (2.5) from the elementary theory, in case of (2.6) from the theory of the double Fourier series given by Hobson [15] or Tonelli [16] and, finally, in the case of (2.7) may presumably be surmised from Hobson's remarks concerning comparative properties of the double and multiple Fourier series in general.

As noted in the introduction the whole problem of multiple harmonic analysis with which we are concerned for rectifiers consists of the evaluation of the multiple Fourier coefficients or the integrals $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ in (2.11), (2.12), and (2.13) as functions of their several parameters. Aside from certain relatively simple closed form solutions obtainable in the one-frequency problem none of these integrals can be evaluated in closed form in terms of elementary functions, except for specialized values of the parameters h, k, k_1, and k_2, but rather in general the best that can be obtained are solutions in a transcendental form of one type or another with the final computations to be done by numerical methods. Although one might wish for more simple results in the two- and three-frequency problems, the power of the methods to be presented should not be underestimated for either theoretical or practical purposes. Finally, in recognition of the original work of Bennett [1, 2] and following previous usage, we term the functions $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ Bennett functions of the νth kind and of multiplicities one, two, and three respectively.

3. Review of results for the one-frequency problem. In the one-frequency problem for the rectifier (2.1) we have for the input the signal (2.2) and for the output the Fourier series (2.5) and have to determine particularly the Fourier coefficients

$$A_m^{(\nu)}(h) = \frac{2}{\pi}\int_{\mathfrak{R}_1} (\cos u - h)^{\nu} \cos mu \, du, \qquad \mathfrak{R}_1 : \cos u \geq h, \quad 0 \leq u \leq \pi, \tag{3.1}$$

as functions of $h = X_0/P$ for fixed $\nu \geq 0$. The variable h takes all real values and, following Kaufman [3], we note three cases of the functions $A_m^{(\nu)}(h)$ according to the scheme: (0) $h \geq 1$, (a) $|h| \leq 1$, and (∞) $h \leq -1$. Clearly case (0) may be disposed of at once for in this case the rectifier is biased so strongly that we have $A_m^{(\nu)}(h) = 0$ for all m.

Consider now cases (a) and (∞) for all $\nu \geq 0$. To begin with, the functions $A_m^{(\nu)}(h)$ satisfy a number of recurrence relations. Thus, for determining the $(\nu + 1)$th kind functions in terms of those of the νth kind we have the recurrence formulas

$$\begin{aligned} \tfrac{1}{2}A_0^{(\nu+1)} &\equiv \tfrac{1}{2}A_1^{(\nu)} - \tfrac{1}{2}hA_0^{(\nu)}, \\ 2mA_m^{(\nu+1)} &\equiv (\nu + 1)A_{m-1}^{(\nu)} - (\nu + 1)A_{m+1}^{(\nu)}, \end{aligned} \tag{3.2}$$

while for purposes of evaluating the $(m + 1)$th order functions in terms of those of the mth and lower orders we have the recurrence formula

$$(m + \nu + 1)A_{m+1}^{(\nu)} \equiv 2mhA_m^{(\nu)} - (m - \nu - 1)A_{m-1}^{(\nu)}, \tag{3.3}$$

where in each of these relations $m \geq 1$. For proofs we refer to Kaufman [3]. For actual evaluation of the functions $A_m^{(\nu)}(h)$ two general solutions are available. In particular, in case (a) Bennett [2] has given the formula

$$A_m^{(\nu)}(h) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 3/2)} (1 - h)^{\nu+1/2} F[\tfrac{1}{2} + m, \tfrac{1}{2} - m; \nu + \tfrac{3}{2}; \tfrac{1}{2}(1 - h)], \tag{3.4}$$

while in case (∞) Lampard [5] has given the result

$$A_m^{(\nu)}(h) = \frac{2^{1-m}}{m!}\,\frac{\Gamma(\nu+1)}{\Gamma(\nu-m+1)}\,|\,h\,|^{\nu-m}\,F[\tfrac{1}{2}(m-\nu),\,\tfrac{1}{2}(m-\nu+1);\,m+1;\,h^{-2}], \tag{3.5}$$

where in each of these formulas $F(a, b; c; x)$ denotes the Gaussian hypergeometric function and where $m \geq 0$. Various alternative forms of these solutions have also been noted by the same authors, Bennett, in particular, indicating certain special results for integral and half integral values of ν; see also Salzberg [6].

Consider next cases (a) and (∞) for integral values of ν. In view of the recurrence relations (3.2) for determining the functions $A_m^{(\nu+1)}(h)$ in terms of the functions $A_m^{(\nu)}(h)$, it is clear that for such values of ν the solutions may be completely expressed in terms of those for a basic set such as $A_m^{(0)}(h)$ or $A_m(h) = A_m^{(1)}(h)$. Actually we choose the latter as being more important in their own right. In case (a) following Bennett [2] we may simplify (3.4) or integrate directly with Salzberg [6] to obtain the special results

$$\begin{aligned}
\tfrac{1}{2}A_0(h) &= \frac{1}{\pi}\,[(1-h^2)^{1/2} - h\cos^{-1}h], \\
A_1(h) &= \frac{1}{\pi}\,[\cos^{-1}h - h(1-h^2)^{1/2}], \\
A_m(h) &= \frac{1}{m\pi}\left\{\frac{\sin[(m-1)\cos^{-1}h]}{m-1} - \frac{\sin[(m+1)\cos^{-1}h]}{m+1}\right\},
\end{aligned} \tag{3.6}$$

where $m \geq 2$, while in case (∞) we have

$$\tfrac{1}{2}A_0(h) = |\,h\,|, \qquad A_1(h) = 1, \qquad A_m(h) = 0, \tag{3.7}$$

where again $m \geq 2$. These formulas represent an essentially complete solution for the functions $A_m(h) = A_m^{(1)}(h)$. For many purposes, however, the corresponding power series expansions are more illuminating. Expanding the elementary functions in (3.6) and combining terms we have, for example, in case (a) for the first few functions $A_m(h) = A_m^{(1)}(h)$ the series

$$\tfrac{1}{2}A_0(h) = \frac{1}{\pi}\left[1 - \frac{\pi}{2}h + \frac{1}{2}h^2 + \frac{1}{24}h^4 + \frac{1}{80}h^6 + \cdots\right], \tag{3.8}$$

$$A_1(h) = \frac{1}{\pi}\left[\frac{\pi}{2} - 2h + \frac{1}{3}h^3 + \frac{1}{20}h^5 + \frac{1}{56}h^7 + \cdots\right], \tag{3.9}$$

$$A_2(h) = \frac{1}{\pi}\left[\frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots\right], \tag{3.10}$$

$$A_3(h) = \frac{h}{\pi}\left[\frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots\right], \tag{3.11}$$

$$A_4(h) = \frac{1}{\pi}\left[-\frac{2}{15} + h^2 - \frac{5}{4}h^4 + \frac{7}{24}h^6 + \cdots\right], \tag{3.12}$$

$$A_5(h) = \frac{h}{\pi}\left[-\frac{2}{5} + \frac{5}{3}h^2 - \frac{7}{4}h^4 + \frac{3}{8}h^6 + \cdots\right], \tag{3.13}$$

each of which converges uniformly for $|\,h\,| \leq 1$ and, indeed quite rapidly, the terms given above, for example, being sufficient to evaluate the functions $\tfrac{1}{2}A_0(h)$ to $A_5(h)$ to

better than 1×10^{-3} units for $|h| \leq \frac{1}{2}$. Finally, in addition to the above formulas and series for the functions $A_m(h) = A_m^{(1)}(h)$ in cases (a) and (∞) Kaufman [3] has given the reflection relations

$$\tfrac{1}{2}A_0(-h) \equiv \tfrac{1}{2}A_0(h) + h, \quad A_1(-h) \equiv 1 - A_1(h), \quad A_m(-h) \equiv (-1)^m A_m(h), \tag{3.14}$$

for determining the functions of negative bias in terms of those of positive bias where again $m \geq 2$.

For elementary applications and general interest line graphs of the first ten functions $\frac{1}{2}A_0(h) = \frac{1}{2}A_0^{(1)}(h)$ and $A_m(h) = A_m^{(1)}(h)$ are presented in Fig. 1 while for applications of greater accuracy the reader may refer to the tables of these functions mentioned in the introduction.

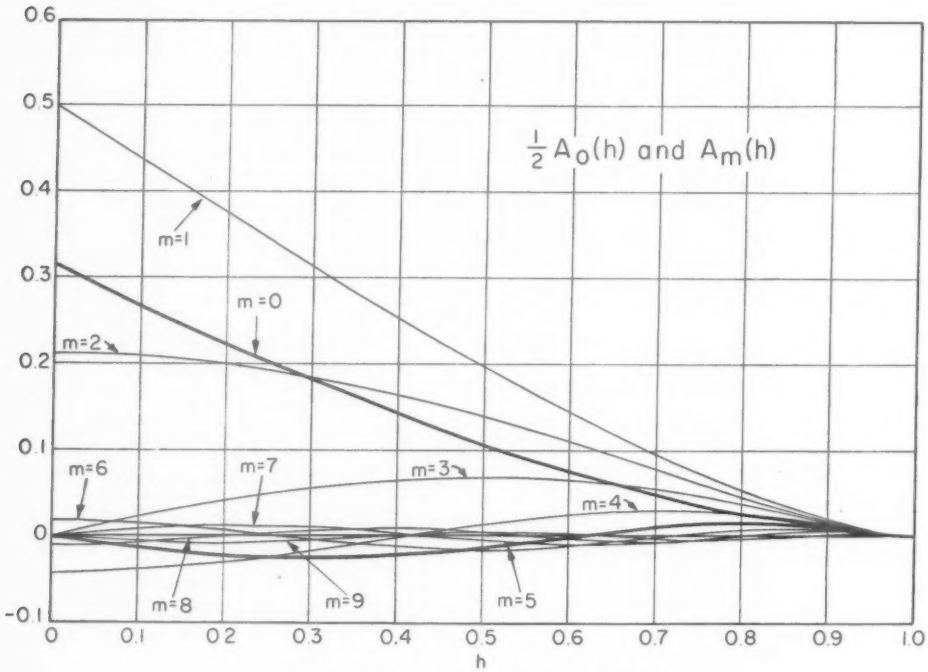

FIG. 1. The functions $\frac{1}{2}A_0(h)$ and $A_m(h, k)$

4. Review of results for the two-frequency problem. In the two-frequency problem for the rectifier (2.1) we have for the input the signal (2.3) and for the output the double Fourier series (2.6) and have to determine particularly the Fourier coefficients

$$A_{mn}^{(\nu)}(h, k) = \frac{2}{\pi^2} \iint_{\mathcal{R}_2} (\cos u + k \cos v - h)^\nu \cos mu \, du \cos nv \, dv, \tag{4.1}$$

$$\mathcal{R}_2 : \cos u + k \cos v \geq h, \qquad 0 \leq u, v \leq \pi,$$

as functions of $h = X_0/P$ and $k = Q/P$ for fixed $\nu \geq 0$. The variable h takes all real values while the variable k ranges over the interval $0 < k \leq 1$ and, as before, we note three cases of the functions $A_{mn}^{(\nu)}(h, k)$ according as: (0) $h \geq 1 + k$, (a) $|h| \leq 1 + k$,

or (∞) $h \leq -1 - k$. Clearly again case (0) may be disposed of at once, the rectifier being biased so strongly that we have $A_{\pm mn}^{(\nu)}(h, k) = 0$ for all m, n while in all cases the functions $A_{\pm mn}^{(\nu)}(h, k)$ are independent of the double sign so that we may drop it hereafter and write simply $A_{mn}^{(\nu)}(h, k)$.

Consider now cases (a) and (∞) for all $\nu \geq 0$. As before the functions $A_{mn}^{(\nu)}(h, k)$ satisfy a number of recurrence relations. Thus, first of all comparing the definitions (3.1) and (4.1) of the functions $A_m^{(\nu)}(h)$ and $A_{mn}^{(\nu)}(h, k)$ one readily observes the fundamental formulas

$$\begin{aligned} A_{mn}^{(\nu)}(h, k) &\equiv \frac{1}{\pi}\int_0^\pi A_m^{(\nu)}(h - k\cos v)\cos nv\, dv, \\ A_{mn}^{(\nu)}(h, k) &\equiv \frac{k^\nu}{\pi}\int_0^\pi A_n^{(\nu)}(h' - k'\cos u)\cos mu\, du, \end{aligned} \tag{4.2}$$

where $h' = h/k$ and $k' = 1/k$, which express the $A_{mn}^{(\nu)}(h, k)$ in terms of the $A_m^{(\nu)}(h)$ and, hence, in a sense constitute a complete solution of the two-frequency problem in terms of the solutions of the one-frequency problem. Next, for determining the $(\nu + 1)$th kind of functions in terms of those of the νth kind we have much as before the recurrence formulas

$$\text{(a)} \qquad \tfrac{1}{2}A_{00}^{(\nu+1)} \equiv \tfrac{1}{2}A_{10}^{(\nu)} + \tfrac{1}{2}kA_{01}^{(\nu)} - \tfrac{1}{2}hA_{00}^{(\nu)},$$

$$\text{(b)} \qquad 2mA_{mn}^{(\nu+1)} \equiv (\nu + 1)A_{m-1,n}^{(\nu)} - (\nu + 1)A_{m+1,n}^{(\nu)}, \tag{4.3}$$

$$\text{(c)} \qquad 2nA_{mn}^{(\nu+1)} \equiv (\nu + 1)kA_{m,n-1}^{(\nu)} - (\nu + 1)kA_{m,n+1}^{(\nu)},$$

where in (b) $m \geq 1$, $n \geq 0$, and in (c) $m \geq 0$, $n \geq 1$, while for purposes of evaluating the $(m + n + 1)$th order functions in terms of those of the $(m + n)$th and lower orders we have the recurrence formulas

$$\begin{aligned} \text{(a)} \qquad & (m + n + \nu + 1)A_{m+1,n}^{(\nu)} \equiv 2mhA_{mn}^{(\nu)} \\ & \qquad - 2mkA_{m,n-1}^{(\nu)} - (m - n - \nu - 1)A_{m-1,n}^{(\nu)}, \\ \text{(b)} \qquad & (m - n + \nu + 3)A_{m+2,n-1}^{(\nu)} \equiv 2(m + 1)hA_{m+1,n-1}^{(\nu)} \\ & \qquad - 2(m + 1)kA_{m+1,n}^{(\nu)} - (m + n - \nu - 1)A_{m,n-1}^{(\nu)}, \\ \text{(c)} \qquad & (n + m + \nu + 1)A_{m,n+1}^{(\nu)} \equiv 2nh'A_{mn}^{(\nu)} \\ & \qquad - 2nk'A_{m-1,n}^{(\nu)} - (n - m - \nu - 1)A_{m,n-1}^{(\nu)}, \\ \text{(d)} \qquad & (n - m + \nu + 3)A_{m-1,n+2}^{(\nu)} \equiv 2(n + 1)h'A_{m-1,n+1}^{(\nu)} \\ & \qquad - 2(n + 1)k'A_{m,n+1}^{(\nu)} - (n + m - \nu - 1)A_{m-1,n}^{(\nu)}, \end{aligned} \tag{4.4}$$

where $h' = h/k$ and $k' = 1/k$ and where in (a) and (c) $m \geq 1$, $n \geq 1$, in (b) $m \geq 0$, $n \geq 1$, and in (d) $m \geq 1$, $n \geq 0$. The proofs of these relations, though computationally lengthy, are entirely elementary and may be effected by combined use of the recurrence relations (3.2) and (3.3) for the functions $A_m^{(\nu)}(h)$ and the formulas (4.2) for expressing the functions $A_{mn}^{(\nu)}(h, k)$ in terms of the functions $A_m^{(\nu)}(h)$; see also Kaufman [4]. For actual evaluation of the functions $A_{mn}^{(\nu)}(h, k)$ only one general solution seems to be available. Thus, in case (∞) Lampard [5] has given the formula

$$\begin{aligned} A_{mn}^{(\nu)}(h, k) = \frac{2^{1-m-n}}{m!n!}\,\frac{\Gamma(\nu + 1)}{\Gamma(\nu - m - n + 1)}\,|\,h\,|^{\nu-m-n}\,k^n \\ \cdot F_4[\tfrac{1}{2}(m + n - \nu), \tfrac{1}{2}(m + n - \nu + 1); m + 1, n + 1; h^{-2}, k^2h^{-2}], \end{aligned} \tag{4.5}$$

where $F_4(a, b; c, d; x, y)$ denotes Appell's fourth type of hypergeometric function of two variables [17] and where $m \geq 0$, $n \geq 0$. Other forms of this solution have also been noted by the same writer.

Consider next cases (a) and (∞) for integral values of ν. Similarly as before the recurrence relations (4.3) for determining the functions $A_{mn}^{(\nu+1)}(h, k)$ in terms of the functions $A_{mn}^{(\nu)}(h, k)$ make it clear that for such values of ν the solutions may be completely expressed in terms of those for a basic set such as $A_{mn}^{(0)}(h, k)$ or $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ and again we make the latter choice. Even for these functions it is not possible to give a complete solution free from integral signs or numerical procedures in case (a). However, in the subcase of case (a) in which $|h| + k \leq 1$, using a method amounting to substitution of the series (3.8) to (3.11) and so on for the functions $A_m(h) = A_m^{(1)}(h)$ into the formulas (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$, Sternberg and Kaufman [7] have derived double power series expansions of the forms

$$\frac{1}{2} A_{00}(h, k) = -\frac{1}{2} h + \frac{1}{\pi} \left\{ \begin{array}{l} 1 + \frac{1}{2} h^2 + \frac{1}{24} h^4 + \frac{1}{80} h^6 + \cdots \\ + \frac{1}{4} k^2 + \frac{1}{8} h^2 k^2 + \frac{3}{32} h^4 k^2 + \cdots \\ + \frac{1}{64} k^4 + \frac{9}{128} h^2 k^4 + \cdots \\ + \frac{1}{256} k^6 + \cdots \\ \cdots \end{array} \right\}, \tag{4.6}$$

$$A_{10}(h, k) = \frac{1}{2} + \frac{h}{\pi} \left\{ \begin{array}{l} -2 + \frac{1}{3} h^2 + \frac{1}{20} h^4 + \frac{1}{56} h^6 + \cdots \\ + \frac{1}{2} k^2 + \frac{1}{4} h^2 k^2 + \frac{3}{16} h^4 k^2 + \cdots \\ + \frac{3}{32} k^4 + \frac{15}{64} h^2 k^4 + \cdots \\ + \frac{5}{128} k^6 + \cdots \\ \cdots \end{array} \right\}, \tag{4.7}$$

$$A_{01}(h, k) = \frac{k}{2} + \frac{hk}{\pi} \left\{ \begin{array}{l} -1 - \frac{1}{6} h^2 - \frac{3}{40} h^4 - \frac{5}{112} h^6 - \cdots \\ - \frac{1}{8} k^2 - \frac{3}{16} h^2 k^2 - \frac{15}{64} h^4 k^2 - \cdots \\ - \frac{3}{64} k^4 - \frac{25}{128} h^2 k^4 - \cdots \\ - \frac{25}{1024} k^6 - \cdots \\ \cdots \end{array} \right\}, \tag{4.8}$$

$$A_{20}(h,k) = \frac{1}{\pi}\left\{\begin{array}{l} \frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots \\ -\frac{1}{2}k^2 + \frac{3}{4}h^2k^2 + \frac{5}{16}h^4k^2 + \cdots \\ +\frac{3}{32}k^4 + \frac{15}{64}h^2k^4 + \cdots \\ +\frac{5}{384}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.9}$$

$$A_{11}(h,k) = \frac{k}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{2}h^2 - \frac{1}{8}h^4 - \frac{1}{16}h^6 - \cdots \\ -\frac{1}{8}k^2 - \frac{3}{16}h^2k^2 - \frac{15}{16}h^4k^2 - \cdots \\ -\frac{1}{64}k^4 - \frac{15}{128}h^2k^4 - \cdots \\ -\frac{5}{1024}k^6 - \cdots \\ \cdots \end{array}\right\}, \tag{4.10}$$

$$A_{02}(h,k) = \frac{k^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{8}h^2 + \frac{3}{32}h^4 + \frac{5}{64}h^6 + \cdots \\ +\frac{1}{48}k^2 + \frac{3}{32}h^2k^2 + \frac{25}{128}h^4k^2 + \cdots \\ +\frac{3}{512}k^4 + \frac{75}{1024}h^2k^4 + \cdots \\ +\frac{5}{2048}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.11}$$

$$A_{30}(h,k) = \frac{h}{\pi}\left\{\begin{array}{l} \frac{2}{3} - h^2 + \frac{1}{4}h^4 + \frac{1}{24}h^6 + \cdots \\ -\frac{3}{2}k^2 + \frac{5}{4}h^2k^2 + \frac{7}{16}h^4k^2 + \cdots \\ +\frac{15}{32}k^4 + \frac{35}{64}h^2k^4 + \cdots \\ +\frac{35}{384}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.12}$$

$$A_{21}(h, k) = \frac{hk}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{2}h^2 - \frac{1}{8}h^4 - \frac{1}{16}h^6 - \cdots \\ -\frac{3}{8}k^2 - \frac{5}{16}h^2k^2 - \frac{21}{64}h^4k^2 - \cdots \\ -\frac{5}{64}k^4 - \frac{35}{128}h^2k^4 - \cdots \\ -\frac{35}{1024}k^6 - \cdots \\ \cdots \end{array}\right\}, \tag{4.13}$$

$$A_{12}(h, k) = \frac{hk^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{8}h^2 + \frac{3}{32}h^4 + \frac{5}{64}h^6 + \cdots \\ +\frac{1}{16}k^2 + \frac{5}{32}h^2k^2 + \frac{35}{128}h^4k^2 + \cdots \\ +\frac{15}{512}k^4 + \frac{175}{1024}h^2k^4 + \cdots \\ +\frac{35}{2048}k^6 + \cdots \\ \cdots \end{array}\right\}, \tag{4.14}$$

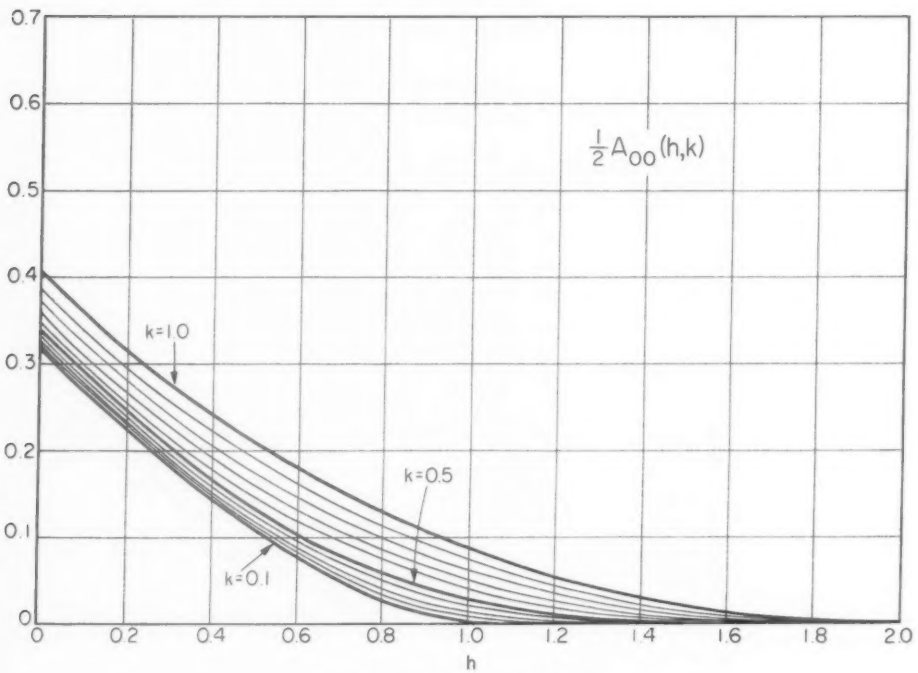

FIG. 2. The function $\frac{1}{2} A_{00}(h, k)$

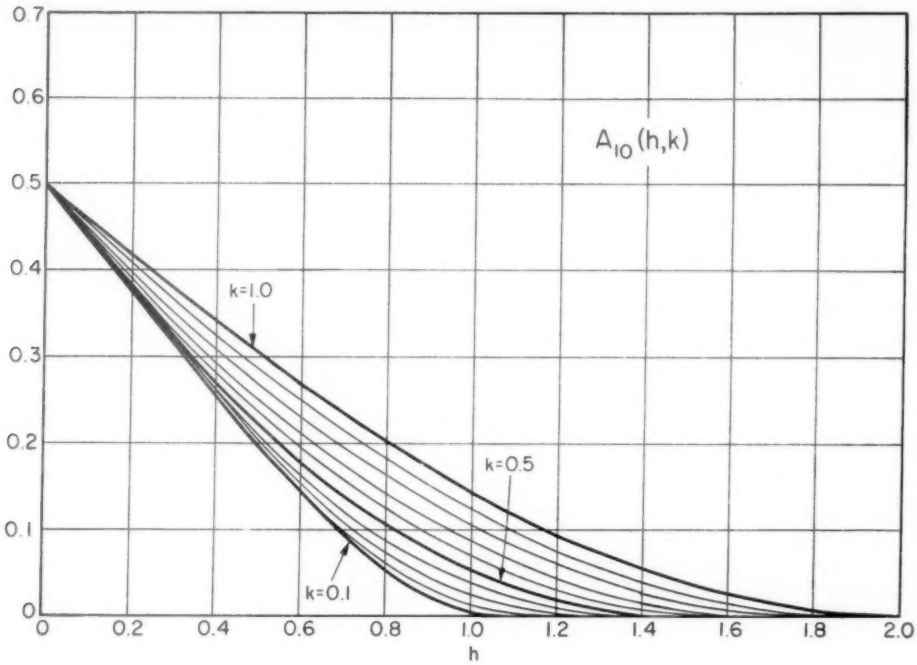

FIG. 3. The function $A_{10}(h, k)$

$$A_{03}(h, k) = \frac{hk^3}{\pi}\left\{\begin{array}{l} -\frac{1}{24} - \frac{1}{16}h^2 - \frac{5}{64}h^4 - \frac{35}{384}h^6 - \cdots \\ -\frac{3}{128}k^2 - \frac{25}{256}h^2k^2 - \frac{245}{1024}h^4k^2 - \cdots \\ -\frac{15}{1024}k^4 - \frac{1445}{12288}h^2k^4 - \cdots \\ -\frac{245}{24576}k^6 - \cdots \\ \cdots \end{array}\right\}, \quad (4.15)$$

and so forth, each of which converges uniformly for $| h | + k \leq 1$ and, in fact, quite rapidly, the terms given above, for example, being sufficient to evaluate the functions $\frac{1}{2}A_{00}(h, k)$ to $A_{03}(h, k)$ to better than 1×10^{-3} units for $| h | + k \leq \frac{1}{2}$. In case (∞) as before we have

$$\begin{aligned} \tfrac{1}{2}A_{00}(h, k) &= | h |, & A_{10}(h, k) &= 1, \\ A_{01}(h, k) &= k, & A_{mn}(h, k) &= 0, \end{aligned} \quad (4.16)$$

where $m + n \geq 2$. For half-integral values of ν in the sub-case of case (a) in which $| h | + k \leq 1$ similar double power series expansions for the functions $A^{(\nu)}_{mn}(h, k)$ also may be obtained by combined application of the formulas (3.4) for the functions $A^{(\nu)}_{m}(h)$

and (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$, the half-integral property of ν making possible the explicit computation of the coefficients in the series, while in case (∞) the formula (4.5) for the functions $A_{mn}^{(\nu)}(h, k)$ also applies with some slight simplifications of the gamma functions. Finally, in addition to the above formulas and series for the functions $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ in cases (a) and (∞) Bennett [2] and Sternberg and Kaufman [7] have given the reflection relations

$$\begin{aligned} \tfrac{1}{2}A_{00}(-h, k) &\equiv \tfrac{1}{2}A_{00}(h, k) + h, & A_{10}(-h, k) &\equiv 1 - A_{10}(h, k), \\ A_{01}(-h, k) &\equiv k - A_{01}(h, k), & A_{mn}(-h, k) &\equiv (-1)^{m+n}A_{mn}(h, k), \end{aligned} \tag{4.17}$$

for determining the functions of negative bias in terms of those of positive bias where again $m + n \geq 2$. These formulas also may be derived in an elementary manner with the aid of previous results for the functions $A_m(h) = A_m^{(1)}(h)$ and the formulas (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$.

For elementary applications and general usefulness line graphs of the first fifteen functions $\frac{1}{2}A_{00}(h, k) = \frac{1}{2}A_{00}^{(1)}(h, k)$ and $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ are presented in Figs. 2 to 16 while for applications requiring greater accuracy the reader as before may refer to the tables of these functions described in the introduction.

5. Some new results for the three-frequency problem. In the three-frequency problem for the rectifier (2.1) we have for the input the signal (2.4) and for the output the triple Fourier series (2.7) and have now to determine the Fourier coefficients

$$A_{**mnl}^{(\nu)}(h, k_1, k_2) = \frac{2}{\pi^3} \iiint_{\mathfrak{R}_3} (\cos u + k_1 \cos v + k_2 \cos w - h)^{\nu} \cdot \cos mu\, du \cos nv\, dv \cos lw\, dw, \tag{5.1}$$

$$\mathfrak{R}_3 : \cos u + k_1 \cos v + k_2 \cos w \geq h, \qquad 0 \leq u, v, w \leq \pi,$$

as functions of $h = X_0/P$, $k_1 = Q/P$, and $k_2 = R/P$ for fixed $\nu \geq 0$. The variable h again takes all real values while the variables k_1 and k_2 range over the intervals $0 < k_2 \leq k_1 \leq 1$ and, as before, there are three cases of the functions $A_{**mnl}^{(\nu)}(h, k_1, k_2)$ according as: (0) $h \geq 1 + k_1 + k_2$, (a) $|h| \leq 1 + k_1 + k_2$, or (∞) $h \leq -1 - k_1 - k_2$. Similarly as before in case (0) we have $A_{**mnl}^{(\nu)}(h, k_1, k_2) = 0$ for all m, n, l while in all cases the functions $A_{**mnl}^{(\nu)}(h, k_1, k_2)$ are independent of the double signs so that we may write simply $A_{mnl}^{(\nu)}(h, k_1, k_2)$.

Consider now cases (a) and (∞) for all $\nu \geq 0$. As before the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$ satisfy a number of recurrence relations. Thus, to begin with comparing the definitions (4.1) and (5.1) of the functions $A_{mn}^{(\nu)}(h, k)$ and $A_{mnl}^{(\nu)}(h, k_1, k_2)$, one readily observes in a manner similar to that previously used the fundamental formulas

$$\begin{aligned} A_{mnl}^{(\nu)}(h, k_1, k_2) &\equiv \frac{1}{\pi}\int_0^{\pi} A_{mn}^{(\nu)}(h - k_2 \cos w, k_1) \cos lw\, dw, \\ A_{mnl}^{(\nu)}(h, k_1, k_2) &\equiv \frac{1}{\pi}\int_0^{\pi} A_{ml}^{(\nu)}(h - k_1 \cos v, k_2) \cos nv\, dv, \\ A_{mnl}^{(\nu)}(h, k_1, k_2) &\equiv \frac{k_1^{\nu}}{\pi}\int_0^{\pi} A_{nl}^{(\nu)}(h_1' - k_1' \cos u, k_1'k_2) \cos mu\, du, \end{aligned} \tag{5.2}$$

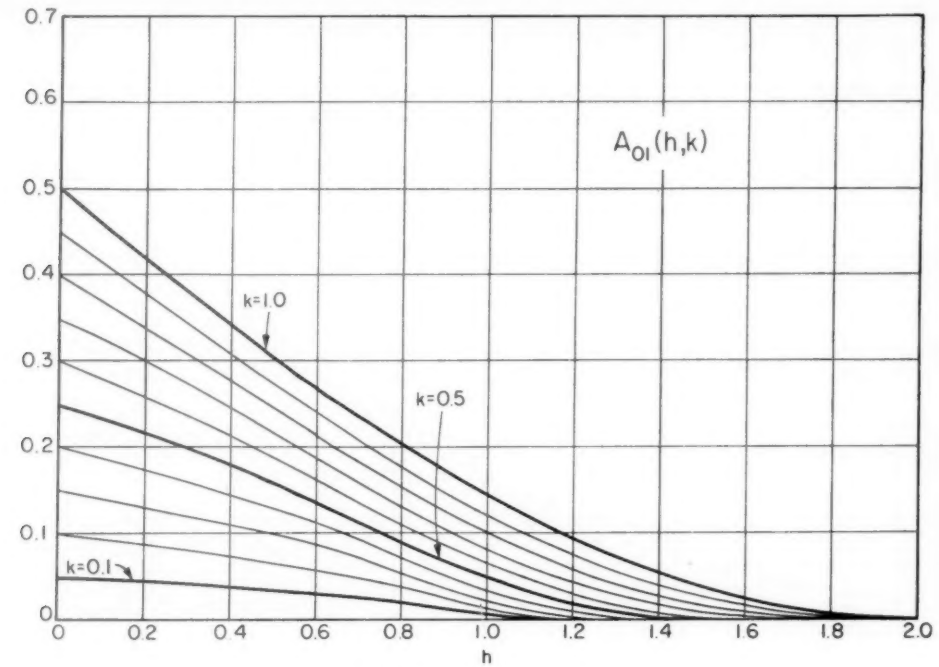

FIG. 4. The function $A_{01}(h, k)$

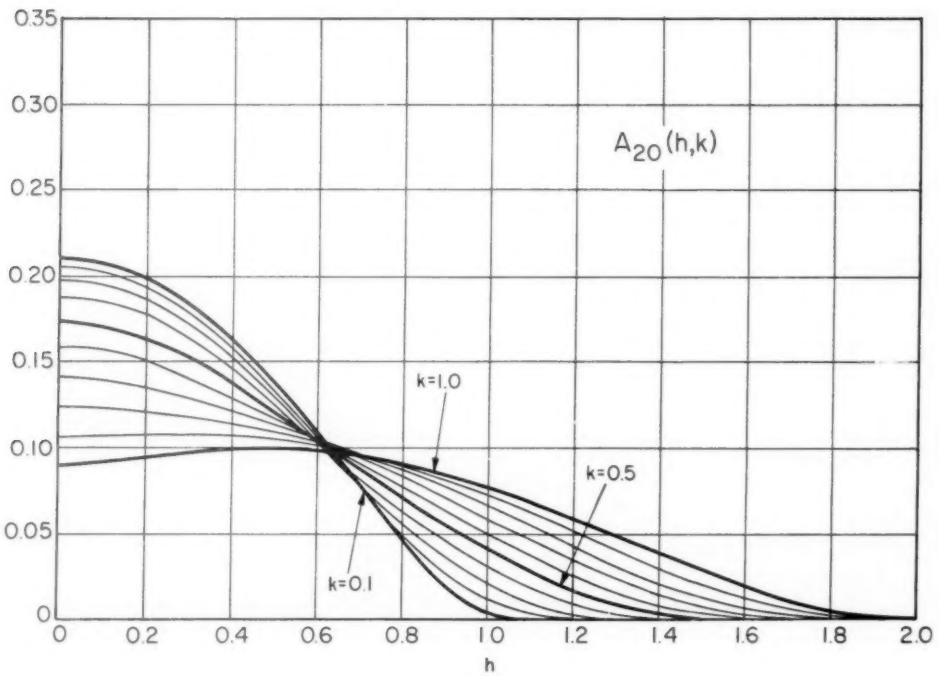

Fig. 5. The function $A_{20}(h, k)$

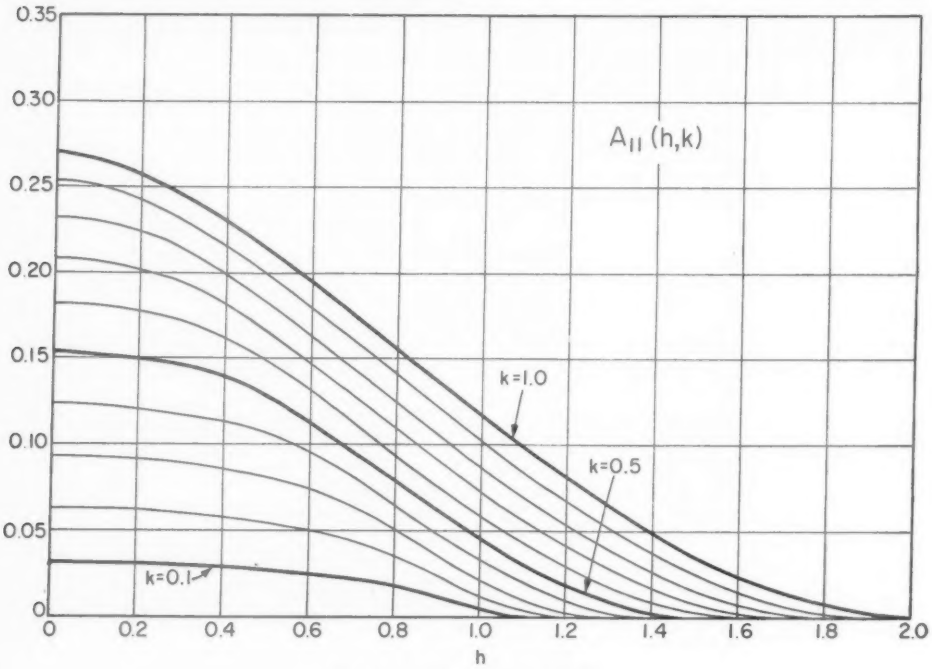

Fig. 6. The function $A_{11}(h, k)$

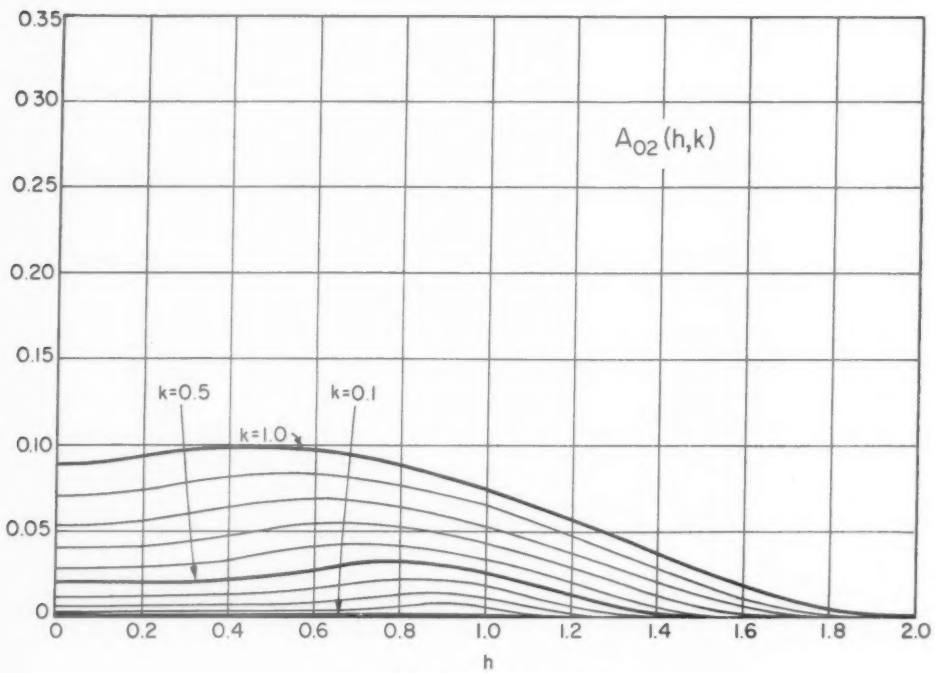

FIG. 7. The function $A_{02}(h, k)$

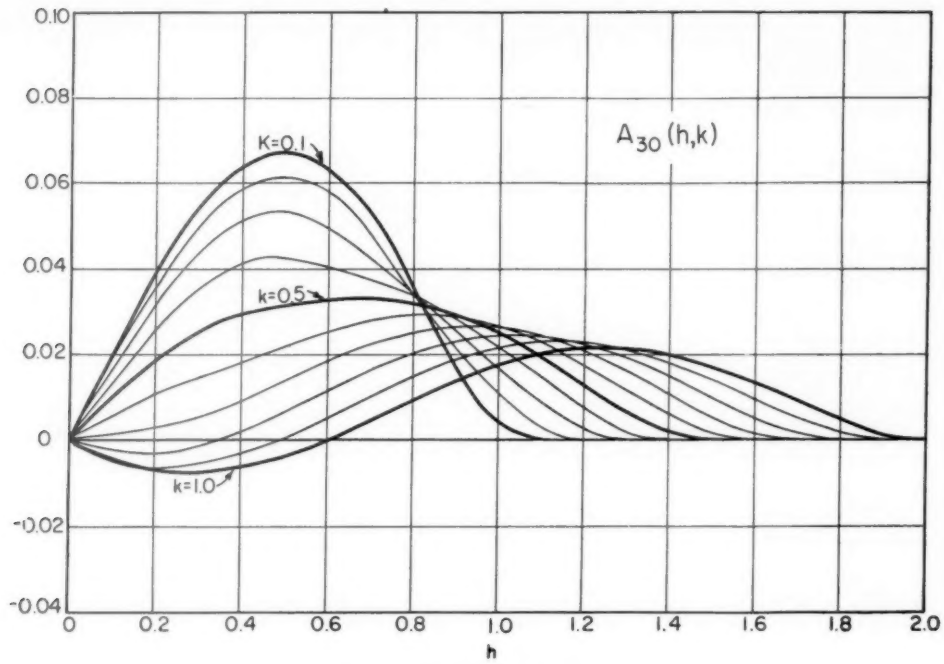

FIG. 8. The function $A_{30}(h, k)$

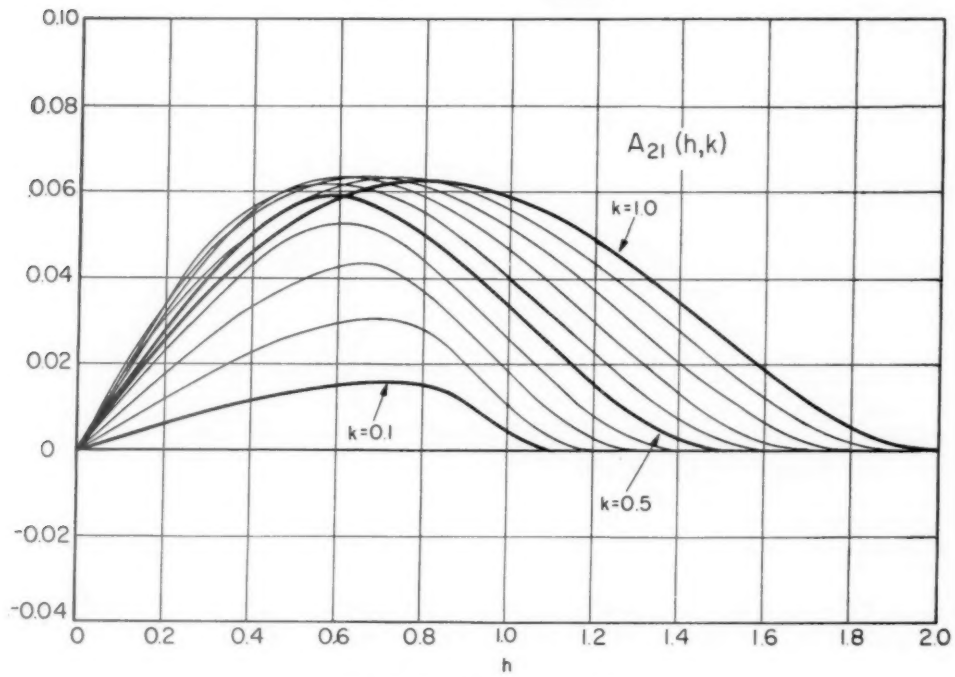

FIG. 9. The function $A_{21}(h, k)$

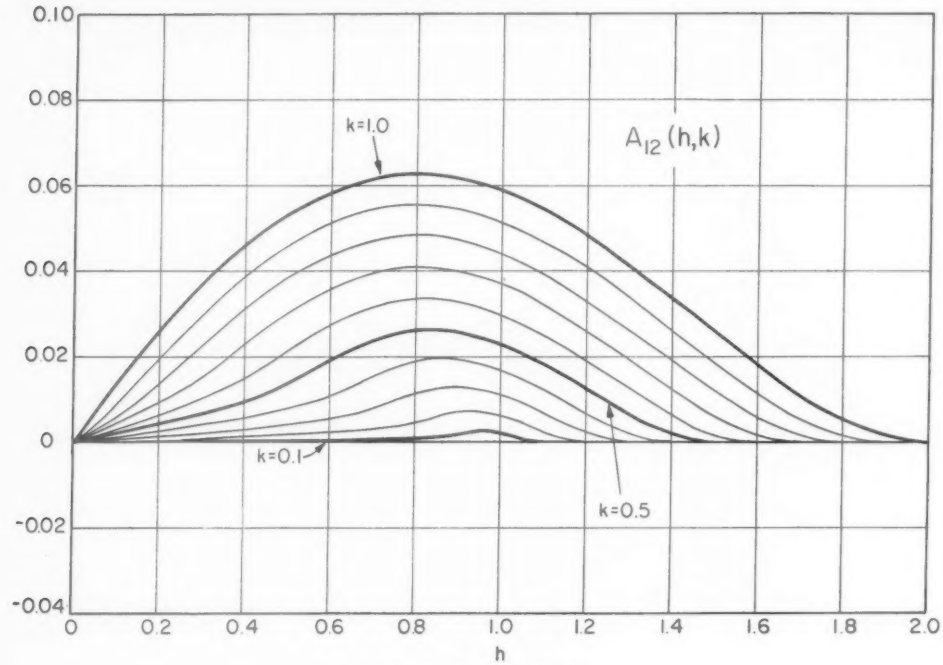

FIG. 10. The function $A_{12}(h, k)$

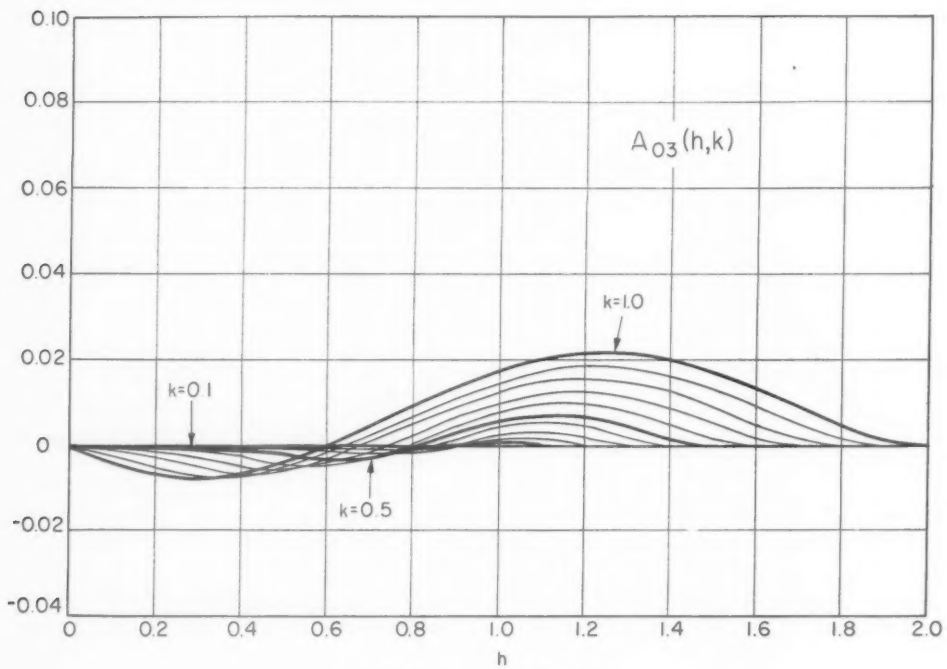

FIG. 11. The function $A_{03}(h, k)$

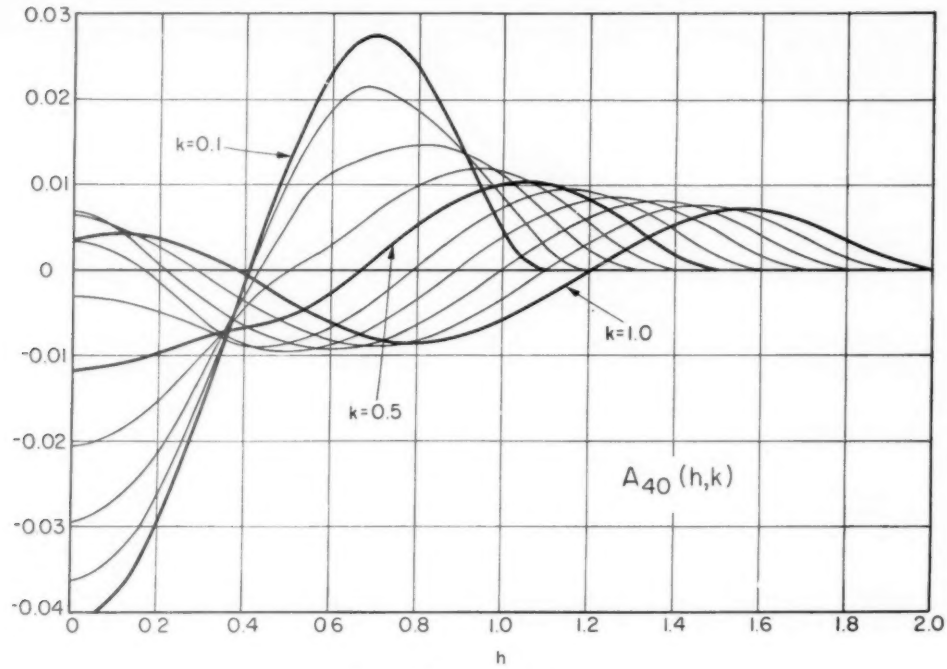

FIG. 12. The function $A_{40}(h, k)$

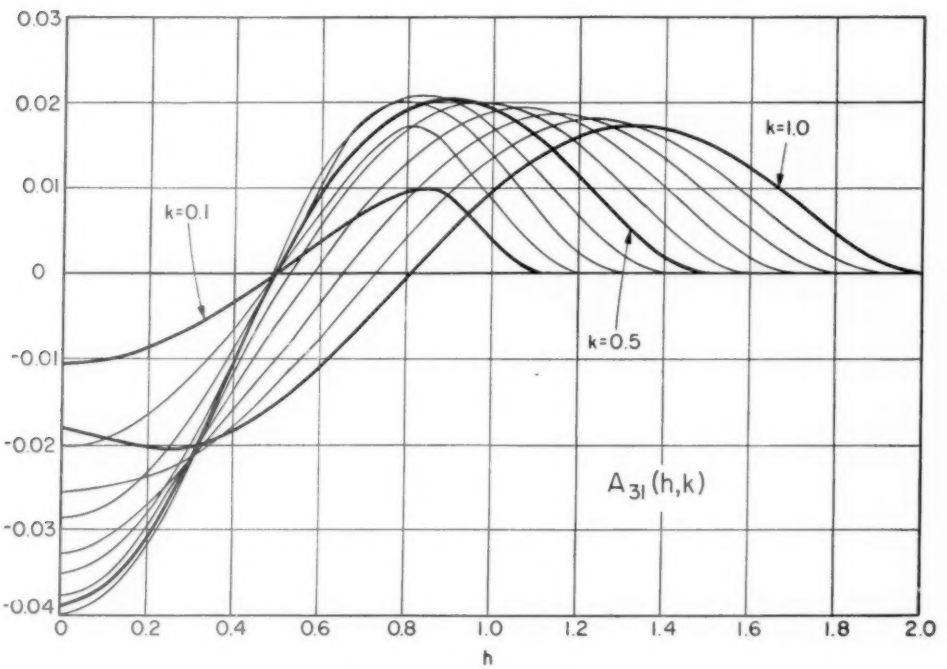

FIG. 13. The function $A_{31}(h, k)$

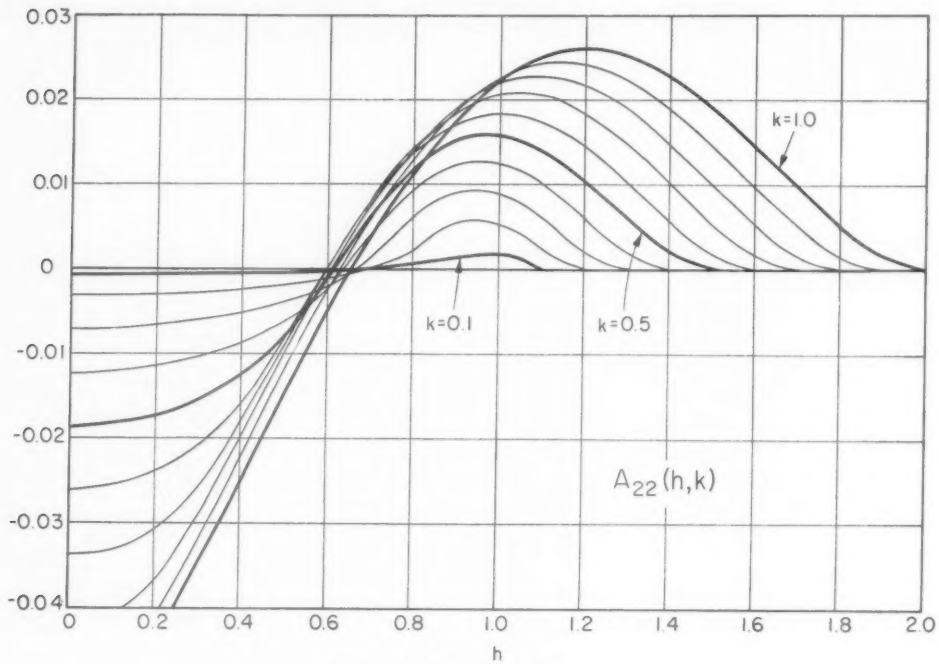

FIG. 14. The function $A_{22}(h, k)$

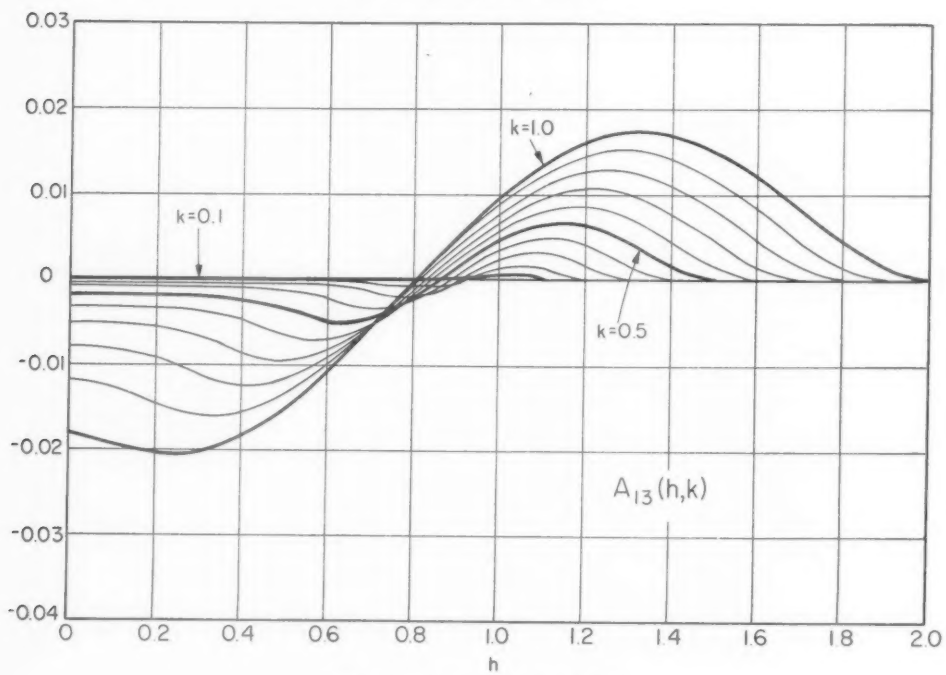

Fig. 15. The function $A_{13}(h, k)$

where $h_1' = h/k_1$ and $k_1' = 1/k_1$, which express the $A_{mnl}^{(\nu)}(h, k_1, k_2)$ in terms of the $A_{mn}^{(\nu)}(h, k)$ and, hence, in a sense constitute a complete solution of the three-frequency problem in terms of the solutions of the two-frequency problem. Next, for determining the higher kind and higher order functions in terms of those of the νth kind and $(m + n + l)$th order we have respectively the recurrence formulas

$$
\begin{aligned}
&\text{(a)} && \tfrac{1}{2}A_{000}^{(\nu+1)} \equiv \tfrac{1}{2}A_{100}^{(\nu)} + \tfrac{1}{2}k_1A_{010}^{(\nu)} + \tfrac{1}{2}k_2A_{001}^{(\nu)} - \tfrac{1}{2}hA_{000}^{(\nu)}\,,\\
&\text{(b)} && 2mA_{mnl}^{(\nu+1)} \equiv (\nu+1)A_{m-1,n,l}^{(\nu)} - (\nu+1)A_{m+1,n,l}^{(\nu)}\,,\\
&\text{(c)} && 2nA_{mnl}^{(\nu+1)} \equiv (\nu+1)k_1A_{m,n-1,l}^{(\nu)} - (\nu+1)k_1A_{m,n+1,l}^{(\nu)}\,,\\
&\text{(d)} && 2lA_{mnl}^{(\nu+1)} \equiv (\nu+1)k_2A_{m,n,l-1}^{(\nu)} - (\nu+1)k_2A_{m,n,l+1}^{(\nu)}\,,
\end{aligned}
\tag{5.3}
$$

where in (b) $m \geq 1$, $n \geq 0$, $l \geq 0$, in (c) $m \geq 0$, $n \geq 1$, $l \geq 0$, and in (d) $m \geq 0$, $n \geq 0$, $l \geq 1$, and the recurrence formulas

$$
\begin{aligned}
\text{(a)}\quad (m+n+l+\nu+1)A_{m+1,n,l}^{(\nu)} &\equiv 2mhA_{mnl}^{(\nu)}\\
&\quad - 2mk_1A_{m,n-1,l}^{(\nu)} - 2mk_2A_{m,n,l-1}^{(\nu)}\\
&\quad - (m-n-l-\nu-1)A_{m-1,n,l}^{(\nu)}\,,\\
\text{(b)}\quad (m-n+l+\nu+3)A_{m+2,n-1,l}^{(\nu)} &\equiv 2(m+1)hA_{m+1,n-1,l}^{(\nu)}\\
&\quad - 2(m+1)k_1A_{m+1,n,l}^{(\nu)} - 2(m+1)k_2A_{m+1,n-1,l-1}^{(\nu)}\\
&\quad - (m+n-l-\nu-1)A_{m,n-1,l}^{(\nu)}\,,\\
\text{(c)}\quad (m+n-l+\nu+3)A_{m+2,n,l-1}^{(\nu)} &\equiv 2(m+1)hA_{m+1,n,l-1}^{(\nu)}\\
&\quad - 2(m+1)k_2A_{m+1,n,l}^{(\nu)} - 2(m+1)k_1A_{m+1,n-1,l-1}^{(\nu)}\\
&\quad - (m-n+l-\nu-1)A_{m,n,l-1}^{(\nu)}\,,\\
\text{(d)}\quad (m-n-l+\nu+5)A_{m+3,n-1,l-1}^{(\nu)} &\equiv 2(m+2)hA_{m+2,n-1,l-1}^{(\nu)}\\
&\quad - 2(m+2)k_2A_{m+2,n-1,l}^{(\nu)} - 2(m+2)k_1A_{m+2,n,l-1}^{(\nu)}\\
&\quad - (m+n+l-\nu-1)A_{m+1,n-1,l-1}^{(\nu)}\,,
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
\text{(a)}\quad (n+l+m+\nu+1)A_{m,n+1,l}^{(\nu)} &\equiv 2nh_1'A_{mnl}^{(\nu)}\\
&\quad - 2nk_1'k_2A_{m,n,l-1}^{(\nu)} - 2nk_1'A_{m-1,n,l}^{(\nu)}\\
&\quad - (n-l-m-\nu-1)A_{m,n-1,l}^{(\nu)}\,,\\
\text{(b)}\quad (n-l+m+\nu+3)A_{m,n+2,l-1}^{(\nu)} &\equiv 2(n+1)h_1'A_{m,n+1,l-1}^{(\nu)}\\
&\quad - 2(n+1)k_1'k_2A_{m,n+1,l}^{(\nu)} - 2(n+1)k_1'A_{m-1,n+1,l-1}^{(\nu)}\\
&\quad - (n+l-m-\nu-1)A_{m,n,l-1}^{(\nu)}\,,\\
\text{(c)}\quad (n+l-m+\nu+3)A_{m-1,n+2,l}^{(\nu)} &\equiv 2(n+1)h_1'A_{m-1,n+1,l}^{(\nu)}\\
&\quad - 2(n+1)k_1'A_{m,n+1,l}^{(\nu)} - 2(n+1)k_1'k_2A_{m-1,n+1,l-1}^{(\nu)}\\
&\quad - (n-l+m-\nu-1)A_{m-1,n,l}^{(\nu)}\,,\\
\text{(d)}\quad (n-l-m+\nu+5)A_{m-1,n+3,l-1}^{(\nu)} &\equiv 2(n+2)h_1'A_{m-1,n+2,l-1}^{(\nu)}\\
&\quad - 2(n+2)k_1'A_{m,n+2,l-1}^{(\nu)} - 2(n+2)k_1'k_2A_{m-1,n+2,l}^{(\nu)}\\
&\quad - (n+l+m-\nu-1)A_{m-1,n+1,l-1}^{(\nu)}\,,
\end{aligned}
\tag{5.5}
$$

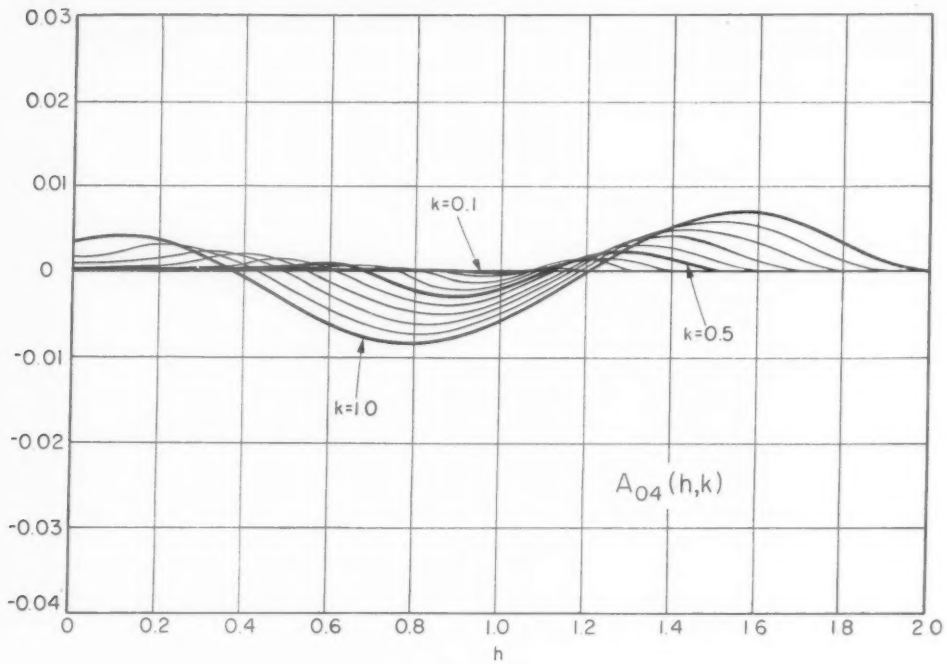

FIG. 16. The function $A_{04}(h, k)$

and

$$\text{(a)}\quad (l+m+n+\nu+1)A^{(\nu)}_{m,n,l+1} \equiv 2lh'_2A^{(\nu)}_{m,n,l} - 2lk'_2A^{(\nu)}_{m-1,n,l} - 2lk_1k'_2A^{(\nu)}_{m,n-1,l} - (l-m-n-\nu-1)A^{(\nu)}_{m,n,l-1},$$

$$\text{(b)}\quad (l-m+n+\nu+3)A^{(\nu)}_{m-1,n,l+2} \equiv 2(l+1)h'_2A^{(\nu)}_{m-1,n,l+1} - 2(l+1)k'_2A^{(\nu)}_{m,n,l+1} - 2(l+1)k_1k'_2A^{(\nu)}_{m-1,n-1,l+1} - (l+m-n-\nu-1)A^{(\nu)}_{m-1,n,l},$$

$$\text{(c)}\quad (l+m-n+\nu+3)A^{(\nu)}_{m,n-1,l+2} \equiv 2(l+1)h'_2A^{(\nu)}_{m,n-1,l+1} - 2(l+1)k_1k'_2A^{(\nu)}_{m,n,l+1} - 2(l+1)k'_2A^{(\nu)}_{m-1,n-1,l+1} - (l-m+n-\nu-1)A^{(\nu)}_{m,n-1,l},$$

$$\text{(d)}\quad (l-m-n+\nu+5)A^{(\nu)}_{m-1,n-1,l+3} \equiv 2(l+2)h'_2A^{(\nu)}_{m-1,n-1,l+2} - 2(l+2)k_1k'_2A^{(\nu)}_{m-1,n,l+2} - 2(l+2)k'_2A^{(\nu)}_{m,n-1,l+2} - (l+m+n-\nu-1)A^{(\nu)}_{m-1,n-1,l+1},$$

(5.6)

where $h'_1 = h/k_1$, $h'_2 = h/k_2$, $k'_1 = 1/k_1$, and $k'_2 = 1/k_2$ and where in formulas (5.4)-(a), (5.5)-(a), and (5.6)-(a) $m \geq 1$, $n \geq 1$, $l \geq 1$; in formulas (5.4)-(b) and (c) $m \geq 0$,

$n \geq 1$, $l \geq 1$; in formulas (5.5)–(b) and (c) $m \geq 1$, $n \geq 0$, $l \geq 1$; in formulas (5.6)–(b) and (c) $m \geq 1$, $n \geq 1$, $l \geq 0$ while in formulas (5.4)–(d), (5.5)–(d), and (5.6)–(d) we have respectively $m \geq -1$, $n \geq 1$, $l \geq 1$; $m \geq 1$, $n \geq -1$, $l \geq 1$; and $m \geq 1$, $n \geq 1$, $l \geq -1$. As before the proof of these relations, though computationally lengthy, is entirely elementary and may be carried out by combined application of the recurrence relations (4.3) and (4.4) for the functions $A^{(\nu)}_{mn}(h, k)$ and the formulas (5.2) for expressing the functions $A^{(\nu)}_{mnl}(h, k_1, k_2)$ in terms of the functions $A^{(\nu)}_{mn}(h, k)$. For actual evaluation of the functions $A^{(\nu)}_{mnl}(h, k_1, k_2)$ no general solutions free from integral signs or numerical procedures appear to be known, the formulas (5.2), the Sternberg and Kaufman approximation process [7, 8, 9] and Feuerstein's generalized Weber-Schafheitlin integral representations [12] being the most general and useful points of departure for numerical work.

Consider next cases (a) and (∞) for integral values of ν and for similar reasons as previously, consider particularly, the basic set of functions $A_{mnl}(h, k_1, k_2) = A^{(1)}_{mnl}(h, k_1, k_2)$. Clearly again it is not possible even here to give a complete solution free from integral signs or numerical procedures in case (a). However, in the subcase of case (a) in which $|h| + k_1 + k_2 \leq 1$, using methods similar to those used before of substitution of the series (4.6) to (4.11) and so on for the functions $A_{mn}(h, k) = A^{(1)}_{mn}(h, k)$ into the formulas (5.2) for the functions $A^{(\nu)}_{mnl}(h, k_1, k_2)$, we may again obtain a solution in multiple power series; in particular, for the case of half-wave rectification in which the bias $h = 0$ we have double power series expansions of the forms

$$\frac{1}{2} A_{000}(0, k_1, k_2) = \frac{1}{\pi} \left\{ \begin{array}{l} 1 + \frac{1}{4} k_1^2 + \frac{1}{64} k_1^4 + \frac{1}{256} k_1^6 + \cdots \\ + \frac{1}{4} k_2^2 + \frac{1}{16} k_1^2 k_2^2 + \frac{9}{256} k_1^4 k_2^2 + \cdots \\ + \frac{1}{64} k_2^4 + \frac{9}{256} k_1^2 k_2^4 + \cdots \\ + \frac{1}{256} k_2^6 + \cdots \\ \cdots \end{array} \right\}, \tag{5.7}$$

$$A_{100}(0, k_1, k_2) = \frac{1}{2}, \qquad A_{010}(0, k_1, k_2) = \frac{1}{2} k_1, \qquad A_{001}(0, k_1, k_2) = \frac{1}{2} k_2, \tag{5.8}$$

$$A_{200}(0, k_1, k_2) = \frac{1}{\pi} \left\{ \begin{array}{l} \frac{2}{3} - \frac{1}{2} k_1^2 + \frac{3}{32} k_1^4 + \frac{5}{384} k_1^6 + \cdots \\ - \frac{1}{2} k_2^2 + \frac{3}{8} k_1^2 k_2^2 + \frac{15}{128} k_1^4 k_2^2 + \cdots \\ + \frac{3}{32} k_2^4 + \frac{15}{128} k_1^2 k_2^4 + \cdots \\ + \frac{5}{384} k_2^6 + \cdots \\ \cdots \end{array} \right\}, \tag{5.9}$$

$$A_{110}(0, k_1, k_2) = \frac{k_1}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{8}k_1^2 - \frac{1}{64}k_1^4 - \frac{5}{1024}k_1^6 - \cdots \\ -\frac{1}{4}k_2^2 - \frac{3}{32}k_1^2k_2^2 - \frac{15}{256}k_1^4k_2^2 - \cdots \\ -\frac{3}{64}k_2^4 - \frac{45}{512}k_1^2k_2^4 - \cdots \\ -\frac{5}{256}k_2^6 - \cdots \\ \cdots \end{array}\right\}, \tag{5.10}$$

$$A_{101}(0, k_1, k_2) = \frac{k_2}{\pi}\left\{\begin{array}{l} 1 - \frac{1}{4}k_1^2 - \frac{3}{64}k_1^4 - \frac{5}{256}k_1^6 - \cdots \\ -\frac{1}{8}k_2^2 - \frac{3}{32}k_1^2k_2^2 - \frac{45}{512}k_1^4k_2^2 - \cdots \\ -\frac{1}{64}k_2^4 - \frac{15}{256}k_1^2k_2^4 - \cdots \\ -\frac{5}{1024}k_2^6 - \cdots \\ \cdots \end{array}\right\}, \tag{5.11}$$

$$A_{020}(0, k_1, k_2) = \frac{k_1^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{48}k_1^2 + \frac{3}{512}k_1^4 + \frac{5}{2048}k_1^6 + \cdots \\ +\frac{1}{16}k_2^2 + \frac{3}{64}k_1^2k_2^2 + \frac{75}{2048}k_1^4k_2^2 + \cdots \\ +\frac{9}{256}k_2^4 + \frac{75}{1024}k_1^2k_2^4 + \cdots \\ +\frac{25}{1024}k_2^6 + \cdots \\ \cdots \end{array}\right\}, \tag{5.12}$$

$$A_{011}(0, k_1, k_2) = \frac{k_1k_2}{\pi}\left\{\begin{array}{l} \frac{1}{2} + \frac{1}{16}k_1^2 + \frac{3}{128}k_1^4 + \frac{25}{2048}k_1^6 + \cdots \\ +\frac{1}{16}k_2^2 + \frac{9}{128}k_1^2k_2^2 + \frac{75}{1024}k_1^4k_2^2 + \cdots \\ +\frac{3}{128}k_2^4 + \frac{75}{1024}k_1^2k_2^4 + \cdots \\ +\frac{25}{2048}k_2^6 + \cdots \\ \cdots \end{array}\right\}, \tag{5.13}$$

$$A_{002}(0, k_1, k_2) = \frac{k_2^2}{\pi}\left\{\begin{array}{l} \frac{1}{4} + \frac{1}{16}k_1^2 + \frac{9}{256}k_1^4 + \frac{25}{1024}k_1^6 + \cdots \\ +\frac{1}{48}k_2^2 + \frac{3}{64}k_1^2k_2^2 + \frac{75}{1024}k_1^4k_2^2 + \cdots \\ +\frac{3}{512}k_2^4 + \frac{75}{2048}k_1^2k_2^4 + \cdots \\ +\frac{5}{2048}k_2^6 + \cdots \\ \cdots \end{array}\right\}, \tag{5.14}$$

and so forth, each of which converges uniformly for $k_1 + k_2 \leq 1$, the k_1 and k_2 being positive and, in fact as before, the convergence is quite rapid, the terms given above, for example, being sufficient to evaluate the functions $\frac{1}{2}A_{000}(0, k_1, k_2)$ to $A_{002}(0, k_1, k_2)$ to better than 1×10^{-3} units for $k_1 + k_2 \leq \frac{1}{2}$. In case (∞) as before we have

$$\begin{gathered} \tfrac{1}{2}A_{000}(h, k_1, k_2) = |h|, \qquad A_{100}(h, k_1, k_2) = 1, \qquad A_{010}(h, k_1, k_2) = k_1, \\ A_{001}(h, k_1, k_2) = k_2, \qquad A_{mnl}(h, k_1, k_2) = 0, \end{gathered} \tag{5.15}$$

where $m + n + l \geq 2$. As before for half-integral values of ν in the sub-case of case (a) in which $|h| + k_1 + k_2 \leq 1$ comparable double or triple power series expansions for the functions $A_{mnl}^{(\nu)}(0, k_1, k_2)$ or $A_{mnl}^{(\nu)}(h, k_1, k_2)$ also may be obtained by combined use of the formulas (3.4) for the functions $A_m^{(\nu)}(h)$, formulas (4.2) for the functions $A_{mn}^{(\nu)}(h, k)$, and formulas (5.2) for the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$, the half-integral character of ν again making possible the explicit computation of the coefficients in the series but in case (∞) no special results exist. Finally, in addition to the above formulas and series for the functions $A_{mnl}(h, k_1, k_2) = A_{mnl}^{(1)}(h, k_1, k_2)$ in cases (a) and (∞) we may note the reflection relations

$$\begin{aligned} \tfrac{1}{2}A_{000}(-h, k_1, k_2) &\equiv \tfrac{1}{2}A_{000}(h, k_1, k_2) + h, \\ A_{100}(-h, k_1, k_2) &\equiv 1 - A_{100}(h, k_1, k_2), \\ A_{010}(-h, k_1, k_2) &\equiv k_1 - A_{010}(h, k_1, k_2), \\ A_{001}(-h, k_1, k_2) &\equiv k_2 - A_{001}(h, k_1, k_2), \\ A_{mnl}(-h, k_1, k_2) &\equiv (-1)^{m+n+l}A_{mnl}(h, k_1, k_2), \end{aligned} \tag{5.16}$$

for determining the functions of negative bias in terms of those of positive bias where again $m + n + l \geq 2$. These formulas likewise may be established in an elementary manner by the use of previous results for the functions $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ and the formulas (5.2) for the functions $A_{mnl}^{(\nu)}(h, k_1, k_2)$.

6. Applications of the theory to the computation of power. The theory of the preceding sections concerning the functions $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ and the graphs of the functions $A_m(h) = A_m^{(1)}(h)$ and $A_{mn}(h, k) = A_{mn}^{(1)}(h, k)$ given in this paper not only may be applied to the computation of modulation product amplitudes but also to the computation of the average output power of the rectifier (2.1) when responding to a several-frequency input of the form (2.2), (2.3), or (2.4), provided, in the two- and three-frequency problems, that the input is non-periodic. Thus, applying

the Parseval theorem [15, 16] or Bessel equality for simple or multiple Fourier series and assuming for the moment that the average output power $\mathcal{P}_0$ in each case may be taken to be one-half of the sum of the squares of the corresponding Fourier coefficients, we have in the one-, two-, and three-frequency problems respectively the formulas

$$\mathcal{P}_0 = \frac{1}{4}P^{2\nu}A_0^{(\nu)2}(h) + \frac{1}{2}P^{2\nu}\sum_{m=1}^{\infty}A_m^{(\nu)2}(h) = \frac{1}{2}P^{2\nu}A_0^{(2\nu)}(h), \tag{6.1}$$

$$\mathcal{P}_0 = \frac{1}{4}P^{2\nu}A_{00}^{(\nu)2}(h, k) + \frac{1}{2}P^{2\nu}\sum_{m,n=0}^{\infty *}A_{mn}^{(\nu)2}(h, k) = \frac{1}{2}P^{2\nu}A_{00}^{(2\nu)}(h, k), \tag{6.2}$$

and

$$\begin{aligned}\mathcal{P}_0 = {} & \frac{1}{4}P^{2\nu}A_{000}^{(\nu)2}(h, k_1, k_2) \\ & + \frac{1}{2}P^{2\nu}\sum_{m,n,l=0}^{\infty *}A_{mnl}^{(\nu)2}(h, k_1, k_2) = \frac{1}{2}P^{2\nu}A_{000}^{(2\nu)}(h, k_1, k_2),\end{aligned} \tag{6.3}$$

provided, for the validity of (6.2), that the input frequency ratio p/q in (2.3) is irrational and, for the validity of (6.3), that the input frequency ratios p/q, q/r, and r/p in (2.4) are all irrational or, in short, that the inputs in question are non-periodic. The proofs of these formulas follow at once from the theorems cited while the justification of the assumptions made concerning the definition of average output power will be given in the last section below. We note in passing that by considering separately the leading terms and the summations in the left hand members of (6.1), (6.2), and (6.3) we may separate out the alternating and direct current output power from the average output power without difficulty. Finally, it may be remarked that by applying the Sternberg and Kaufman approximation process [7, 8, 9] to the squared characteristics of more general continuous modulators than rectifiers, output power computations for these more general problems may also be carried out in terms of the same Bennett functions as here.

7. Some concluding remarks. In concluding this paper on multiple harmonic analysis in rectifier problems it seems worthwhile to note that the simple and multiple Fourier series expansions of the rectifier output (2.5), (2.6), and (2.7) are also generalized Fourier series in the sense of Bohr [18, 19, 20] provided, in the two- and three-frequency problems, that the inputs are non-periodic and that $\nu > 0$. The significance of this remark and also its justification lies in the fact that under the conditions noted we have for the functions $A_m^{(\nu)}(h)$, $A_{mn}^{(\nu)}(h, k)$, and $A_{mnl}^{(\nu)}(h, k_1, k_2)$ the further formulas

$$A_m^{(\nu)}(h) = 2\lim_{T\to\infty}\frac{1}{T}\int_0^T Y^{\nu}[\cos(pt + \theta_p); h]\cos(\omega_m t + \phi_m)\,dt, \tag{7.1}$$

$$\begin{aligned}A_{mn}^{(\nu)}(h, k) = {} & 2\lim_{T\to\infty}\frac{1}{T}\int_0^T Y^{\nu}[\cos(pt + \theta_p) \\ & + k\cos(qt + \theta_q); h]\cos(\omega_{mn}t + \phi_{mn})\,dt,\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}A_{mnl}^{(\nu)}(h, k_1, k_2) = {} & 2\lim_{T\to\infty}\frac{1}{T}\int_0^T Y^{\nu}[\cos(pt + \theta_p) \\ & + k_1\cos(qt + \theta_q) + k_2\cos(rt + \theta_r); h]\cos(\omega_{mnl}t + \phi_{mnl})\,dt,\end{aligned} \tag{7.3}$$

where for the validity of (7.2) and (7.3) we thus assume that the frequency ratio p/q in (7.2) and the frequency ratios p/q, q/r, and r/p in (7.3) are all irrational and that $\nu > 0$. To establish these formulas we begin by noting with the aid of Hobson [15] and Tonelli [16] that each of the corresponding Fourier series expansions (2.5), (2.6), and (2.7) converges uniformly for $\nu > 0$. To complete the proof of (7.2), for example, we now multiply (2.6) by $\cos(\omega_{*m'n'}t + \phi_{*m'n'})$ and note that the resulting series also converges uniformly for $\nu > 0$ so that for each $\epsilon > 0$ there exist integers $M > m'$ and $N > n'$ such that the remainder of the new series after the terms

$$P^{\nu} A^{(\nu)}_{*MN}(h, k) \cos(\omega_{*MN}t + \phi_{*MN}) \cos(\omega_{*m'n'}t + \phi_{*m'n'}), \tag{7.4}$$

never exceeds $\frac{1}{2}\epsilon P^{\nu}$ in absolute value regardless of the value of t. Consequently, integrating termwise up to and including the terms (7.4), multiplying through the resulting relation by $2/TP^{\nu}$, and taking limits we may thence interchange the order of limiting and summation operations in this finite part of the series to conclude that

$$\left| A^{(\nu)}_{*m'n'}(h, k) - 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T Y^{\nu}[\cos(pt + \theta_p) + k \cos(qt + \theta_q); h] \cos(\omega_{*m'n'}t + \phi_{*m'n'})\, dt \right| \tag{7.5}$$

$$\leq \frac{2}{P^{\nu}} \lim_{T\to\infty} \frac{1}{T} \int_0^T \tfrac{1}{2}\epsilon P^{\nu}\, dt = \epsilon,$$

all of which operations are permissible and valid provided the ratio p/q is irrational and $\nu > 0$ so that (7.2) is thereby proven. Similar proofs may be given of course for formulas (7.1) and (7.3) also.

Consider now the implications of the formulas (7.1), (7.2), and (7.3). First of all, these formulas applied to the constant terms in the various Fourier series expansions considered, assert that under the stated conditions time averages are equal to phase averages. Secondly, since squaring $Y^{\nu}(X; X_0)$ is equivalent to replacing ν by 2ν, it is now clear that the formulas (6.1), (6.2), and (6.3) previously given under the same conditions for the average output power $\mathcal{P}_0$ were indeed correct, at least for $\nu > 0$. In closing we may note, finally, that in a sense we have here a simplified example of an ergodic situation while the input function and output series are always almost periodic functions in the sense of Bohr.

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ON CURVED SHOCK WAVES IN THREE-DIMENSIONAL GAS FLOWS*

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Introduction. In this paper we discuss the problem of shock waves in three-dimensional steady rotational flows of an ideal gas with viscosity and heat conduction neglected and subject to no extraneous force. If a shock wave arises, e.g., as the result of the presence of an obstacle in the field of the flow, it will be assumed that this divides the flow under consideration into a region 1 and a region 2. We suppose region 1 to be traversed by the gas before contact with the shock surface and region 2 to be that into which gas enters after passing through the shock surface. It is assumed that this surface is given by a continuous and differentiable function of coordinates and that it has a continuous and differentiable unit normal which we suppose directed from region 1 to region 2. Furthermore, we assume the required differentiability conditions to be satisfied by the velocity components u_i , the density ρ, pressure p and entropy S in each region 1 and 2 so that it is possible to express the surface covariant derivatives of these quantities along either side of the shock surface.

If the flow in front of the shock is known then the flow behind the shock is determined by the well-known Rankine-Hugoniot relations. The main object of this paper is to obtain formulas for the determination of the gradients of velocity components, pressure, density and entropy behind the shock surface when the flow in front is known. These formulas have been obtained for plane flows by Thomas [1]. In his analysis Thomas assumed that γ, the ratio of specific heats, is constant throughout the flow. Truesdell [2] has extended these results to fluids obeying an arbitrary equation of state. The same can be done for the three-dimensional gas flows also, although we have carried out the calculations for the case when γ is constant. Truesdell has further put Thomas's results in more compact form by introducing certain dimensionless variables which have been found useful in the following discussion also.

The required derivatives are obtained by differentiating the Rankine-Hugoniot relations along the shock surface. Gauss-Weingarten formulas and various other results of the geometry of surfaces concerning the principal normal curvatures are found useful in the analysis. The complete analysis has been given only for the case of uniform flow upstream of the shock. As anticipated, the flow behind the shock wave is found in general to be rotational. The explicit determination of the vorticity components is carried out. This leads to the formulation of a theorem regarding the characterization of surfaces behind which the flow remains irrotational. It is found that the plane, the right circular cone, the cylinder and the developable helicoid are the only such surfaces. Furthermore, it is found that the component of vorticity along the normal to the shock surface vanishes at every point of the surface—a property shared by the contact discontinuities which

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are surfaces across which the normal component of the velocity is continuous while the tangential components are discontinuous.

Formal methods of tensor analysis have been used throughout the following discussion.

1. Equations of motion. We consider the differential relations governing the steady flow of a perfect gas devoid of viscosity and heat conduction and subject to no extraneous force, namely [3, 4],

$$p_{,i} + \rho u_j u_{i,j} = 0, \qquad \text{(equations of motion)}, \tag{1}$$

$$\rho_{,i} u_i + \rho u_{j,j} = 0, \qquad \text{(equation of continuity)}, \tag{2}$$

in which p, ρ and u_i denote the pressure, density and velocity components respectively. We assume the motion referred to a system of rectangular coordinates x^i; then the comma in the above equations and in the following, represents partial differentiation. It is to be understood in the above and in the following discussion, unless the contrary is stated, that an index which occurs twice in a term is to be summed over the admissable values of the index. Since there is no distinction between covariant and contravariant indices within a rectangular system, we may write any index as a subscript or a superscript without modifying the value of the term in which the index occurs.

In addition there is the equation of state, *viz.* [3, 4],

$$p = \exp(S/Jc_v)\rho^\gamma, \tag{3}$$

where S is the entropy, J the mechanical equivalent of heat and γ is the ratio of two specific heats c_p and c_v assumed constant. The entropy remains constant on a streamline but varies from streamline to streamline i.e., $S_{,i} u_i = 0$. By differentiating (3) and using the Eqs. (1) and (2) we obtain

$$u_{i,j} u_i u_j - c^2 u_{k,k} = 0, \tag{4}$$

where $c^2 = \gamma p/\rho$, is the adiabatic speed of sound.

2. The Rankine-Hugoniot relations. Denote by u_{1i}, ρ_1 and p_1 the velocity components, density and pressure on the side of the wave bordering region 1 and correspondingly by u_{2i}, ρ_2 and p_2 the values of these quantities on the side of the wave bordering region 2. Put

$$u_{1n} = u_{1i}\xi^i, \qquad u_{2n} = u_{2i}\xi^i,$$

so that u_{1n} and u_{2n} are the normal components of the velocity on sides 1 and 2 of the wave surface, respectively. Then the Rankine-Hugoniot relations for the stationary wave under consideration can be written as

$$[u_i] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)\xi_i}{-(\gamma + 1)\rho_1 u_{1n}}, \tag{5}$$

$$[p] = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)}{(\gamma + 1)}, \tag{6}$$

$$[\rho] = \frac{2\rho_1(\rho_1 u_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma - 1)\rho_1 u_{1n}^2}. \tag{7}$$

Here the bracket [ ] denotes the difference of the values on the two sides of the shock surface of the quantity enclosed, e.g., $[u_i] = u_{2i} - u_{1i}$.

3. Coordinate system on the shock surface. In the following analysis it has been found convenient and advantageous to have the lines of curvature as the Gaussian coordinate curves on the shock surface. Denote these coordinates by y^1, y^2. Then the lines of curvature are y^1 = constant and y^2 = constant and the parametric equations of the surface are given by

$$x^i = x^i(y^1, y^2), \qquad i = 1, 2, 3. \tag{8}$$

As is customary we shall use Latin letters for the indices referring to the space variables and Greek letters for the indices referring to the surface variables. Thus the Latin indices will assume values 1, 2 and 3, and Greek indices the values 1 and 2.

The unit surface tangent vectors to the coordinate curves are [5a] $\delta^\alpha_1 g_{11}$ and $\delta^\alpha_2 g_{22}$ respectively where

$$g_{\alpha\beta} = 1/(a_{\alpha\beta})^{1/2},$$

and $a_{\alpha\beta}$ are the components of the first fundamental form of the surface

$$a_{\alpha\beta} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^i}{\partial y^\beta}, \tag{9}$$

while δ is the Kronecker delta. The corresponding space components of the unit tangent vectors are $x^i_1 g_{11}$ and $x^i_2 g_{22}$ respectively, where we have put

$$x^i_\alpha = \frac{\partial x^i}{\partial y^\alpha}. \tag{10}$$

Then x^i_α are the components of a contravariant space vector and a covariant surface vector. We shall call these quantities the components of projection tensor [6]. We have already assumed that the normal vector to the shock surface is directed from region 1 to region 2. We further choose the orientation of the surface normal and Gaussian coordinates in such a way that x^i_1, x^i_2 and ξ^i have the same orientation as the x^1-, x^2- and x^3-axis.

Now consider the velocity field u_i and let ζ_i denote the part of the velocity field which is normal to the surface and let v_i denote that part of u_i which lies in the tangent plane of the surface. Evidently we have

$$u_i = v_i + \zeta_i .$$

By multiplying both sides of this relation by x^i_α we obtain [6]

$$x^i_\alpha u_i = v_\alpha , \tag{11}$$

where we have put $x^i_\alpha v_i = v_\alpha$. We say that the tensor x^i_α projects u_i onto the tangent plane to the surface, $v_\alpha g_{\alpha\alpha}$ (α not summed) being the tangential component of the velocity along the y^α curve.

Following Truesdell [2] we introduce the dimensionless variables:

$$\delta = \frac{2(\rho_1 u_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma - 1)\rho_1 u_{1n}^2}, \qquad \tau_\alpha = \frac{v_\alpha g_{\alpha\alpha}}{u_{1n}},$$
$$\chi_\alpha = \frac{v_\alpha g_{\alpha\alpha}}{u_{2n}}, \qquad M_{1n} = \frac{u_{1n}}{c_1}, \qquad M_{2n} = \frac{u_{2n}}{c_2}, \tag{12}$$

where α is not summed, δ is the shock strength and τ_α are the components of the obliquity of the shock. With these abbreviations the shock relations can be written as

$$[u_i] = \frac{-\delta u_{1n}\xi_i}{1+\delta}, \tag{13}$$

$$[p] = \frac{\delta}{1+\delta}\rho_1 u_{1n}^2, \tag{14}$$

$$[\rho] = \delta\rho_1 . \tag{15}$$

Multiplying both sides of (13) by ξ_i and x_α^i we easily deduce

$$u_{2n} = \frac{u_{1n}}{1+\delta} = \frac{\rho_1 u_{1n}}{\rho_2}, \tag{16}$$

and

$$v_{2\alpha} = v_{1\alpha} . \tag{17}$$

4. Differentiation of the shock relations. Differentiating the relations (5) to (7) with respect to y^1 and y^2 we get relations along the shock surface of the form

$$u_{i,j}x_\alpha^j = u_{1i,j}x_\alpha^j + A_{i\alpha} = A_{i\alpha}^*, \tag{18}$$

$$p_{,j}x_\alpha^j = p_{1,j}x_\alpha^j + B_\alpha = B_\alpha^*, \tag{19}$$

$$\rho_{,j}x_\alpha^j = \rho_{1,j}x_\alpha^j + C_\alpha = C_\alpha^*, \tag{20}$$

where for simplicity we have omitted the subscript 2 on the quantities appearing in the left members of these equations. The explicit formulas for the quantities $A_{i\alpha}$, B_α and C_α are readily obtained by differentiating the right members of the equations (5) to (7), respectively, with the help of the Gauss-Weingarten formulas of differential geometry (see Sec. 6). We see that these quantities are expressible in terms of the known parameters of the flow in the region 1 and their partial derivatives along the shock surface, the projection tensor x_α^i, the surface normal ξ and the two principal curvatures. Following Thomas [1], any quantity which can be so expressed will be said, in the following, to have an *allowable determination.*

5. Derivation of the partial derivatives behind the shock surface. Consider the Eqs. (1), (2), (4), (18), (19), and (20). From this set of six relations we can determine the partial derivatives of the velocity components u_i, the density ρ and the pressure p immediately behind the shock surface. For this purpose we first eliminate the derivatives $p_{,i}$ to obtain the following equations, which we represent as two sets of equations for convenience of reference, namely

$$\rho_{,j}x_\alpha^j = C_\alpha^*, \tag{21}$$

$$\rho_{,j}u_j + \rho u_{k,k} = 0, \tag{22}$$

and

$$u_{i,j}x_\alpha^j = A_{i\alpha}^*, \tag{23}$$

$$\rho u_{i,j}x_\alpha^j u_i = -B_\alpha^*, \tag{24}$$

$$\rho u_{i,j}u_i u_j - \gamma p u_{k,k} = 0. \tag{25}$$

Now from (23) to (25) we can obtain $u_{i,j}$ and then from (21) and (22) $\rho_{,i}$ can be found; after this the Eqs. (1) yield the values of the derivatives $p_{,i}$. To effect the above determination we define a matrix $\| C_{ij} \|$ such that

$$C_{i\alpha} = \lambda_{i\alpha} \; ; \qquad C_{i3} = u_i/u_n \; , \tag{26}$$

where $\lambda_{i\alpha}$ denote the space components of the unit tangent vector to the curve y^α and are given, as remarked above, in terms of the projection tensor by

$$\lambda_{i\alpha} = \lambda^i_\alpha = x^i_\alpha g_{\alpha\alpha} \; , \qquad \text{no summation on } \alpha.$$

The determinant of the matrix $\| C_{ij} \|$ is easily seen to be unity. Hence we can define the quantities D_{ik} by

$$D_{ik}C_{kj} = \delta_{ij} \; ; \qquad D_{ki}C_{jk} = \delta_{ij} \; . \tag{27}$$

The second relation (27) follows from the first relation and conversely. Since we shall have to use the explicit form of the matrix $\| D_{ij} \|$, it can be easily seen to be given by

$$\begin{Vmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{Vmatrix} = \begin{Vmatrix} \dfrac{(\lambda^2_2 u_3 - \lambda^3_2 u_2)}{u_n} & \dfrac{(\lambda^3_1 u_2 - \lambda^2_1 u_3)}{u_n} & \xi_1 \\ \dfrac{(\lambda^3_2 u_1 - \lambda^1_2 u_3)}{u_n} & \dfrac{(\lambda^1_1 u_3 - \lambda^3_1 u_1)}{u_n} & \xi_2 \\ \dfrac{(\lambda^1_2 u_2 - \lambda^2_2 u_1)}{u_n} & \dfrac{(\lambda^2_1 u_1 - \lambda^1_1 u_2)}{u_n} & \xi_3 \end{Vmatrix} . \tag{28}$$

Now define the quantities B_{ij} by

$$B_{ij} = u_{l,m} C_{li} C_{mj} \; . \tag{29}$$

Then from (23), (24), (25) and (29) we have

$$\begin{aligned} B_{\alpha\beta} &= x^i_\alpha A^*_{i\beta} g_{\alpha\alpha} g_{\beta\beta} \; , & B_{\alpha 3} &= -\frac{B^*_\alpha g_{\alpha\alpha}}{\rho u_n} \; , \\ B_{3\alpha} &= \frac{A^*_{i\alpha} u_i g_{\alpha\alpha}}{u_n} \; , & B_{33} &= \frac{u_{k,k}}{M_n^2} \; , \end{aligned} \tag{30}$$

where α, β are not summed.

Furthermore, by multiplying both sides of (29) by $D_{is}D_{jt}$ and using the relation (27) we readily obtain

$$u_{l,m} = B_{ij} D_{il} D_{jm} \; . \tag{31}$$

Now the first eight elements of the matrix $\| B_{ij} \|$ have an allowable determination as can be readily seen. However, B_{33} is given in terms of the u_{kk} ; it therefore remains to determine the quantity B_{33} . But this can easily be done by contracting the indices l and m in (31), *viz.*,

$$u_{k,k} = B_{ij} D_{ik} D_{jk} \; ,$$

which, by the help of the matrix (28), gives the value

$$u_{k,k} = B_{11}(1 + \chi_1^2) + B_{22}(1 + \chi_2^2) + 2B_{(12)}\chi_1\chi_2 - 2B_{(13)}\chi_1 - 2B_{(23)}\chi_2 + B_{33} \; ,$$

where χ's are defined by (12) and for brevity we have introduced

$$B_{(ij)} = \tfrac{1}{2}(B_{ij} + B_{ji}).$$

Thus we arrive at the value of B_{33}

$$B_{33} = \frac{B_{11}(1 + \chi_1^2) + B_{22}(1 + \chi_2^2) + 2B_{(12)}\chi_1\chi_2 - 2B_{(13)}\chi_1 - 2B_{(23)}\chi_2}{M_n^2 - 1}, \qquad (32)$$

which furnishes an allowable determination of B_{33} . Hence the relations (31), in which B_{33} is given by (32), provide an allowable determination of the gradients of velocity components behind the shock wave.

To effect the determination of the gradient of density behind the shock surface we observe that the set of equations (22) and (23) can be written in the form

$$\rho_{,j} C_{ji} = d_i ,$$

where

$$d_1 = g_{11} C_1^* , \qquad d_2 = g_{22} C_2^* ,$$
$$d_3 = -\rho u_{k,k}/u_n = -\rho B_{33} M_n^2/u_n .$$

Hence we have

$$\rho_{,i} = d_i D_{ii} , \qquad (33)$$

as the allowable determination of the quantities $\rho_{,i}$. Finally from (1) the derivatives $p_{,i}$ behind the shock surface can be determined by the equations

$$p_{,i} = -\rho u_j u_{i,j} = -\rho u_n B_{j3} D_{ii} . \qquad (34)$$

6. Calculation of the invariants d_i and B_{ij}. It is possible to give an explicit formulation of the invariants d_i and B_{ij} under the assumption that the flow in front of the shock is uniform, i.e., the velocity, pressure, and density are constant in region 1. To effect this formulation we need some results of the differential geometry which are briefly stated below [5a].

We have already noted the expressions for the first fundamental form of the surface, the projection tensor and the space components of the unit tangent vectors to the coordinate curves. They are

$$a_{\alpha\beta} = x_\alpha^i x_\beta^i , \qquad x_\alpha^i = \frac{\partial x^i}{\partial y^\alpha} , \qquad \lambda_\alpha^i = g_{\alpha\alpha} x_\alpha^i , \qquad \alpha \text{ not summed},$$

respectively. In addition we need the expression for the normal curvatures in the directions of the coordinate curves (which are the lines of curvatures and hence these curvatures are the principal normal curvatures). They are

$$k_1 = b_{11}/a_{11} ; \qquad k_2 = b_{22}/a_{22} , \qquad (35)$$

where $b_{\alpha\beta}$ are the components of the second fundamental form of the surface, i.e.;

$$b_{\alpha\beta} = \tfrac{1}{2}\epsilon^{\gamma\delta} e_{ijk} x_{\alpha;\beta}^i x_\gamma^j x_\delta^k . \qquad (36)$$

In the above relation $\epsilon^{\gamma\delta}$ are the components of the surface permutation tensor with the properties

$$\epsilon^{11} = \epsilon^{22} = 0; \qquad \epsilon^{12} = 1/(a)^{1/2}; \qquad \epsilon^{21} = -1/(a)^{1/2},$$

where a is the det $|| a_{\alpha\beta} ||$, while e_{ijk} are the components of the permutation tensor of the space with the properties

(i) $e_{ijk} = 0$ if any two of the indices i, j and k are equal,

(ii) $e_{123} = e_{231} = e_{312} = +1$,

(iii) $e_{132} = e_{321} = e_{213} = -1$,

and $x^i_{\alpha;\beta}$ denotes the surface covariant differentiation of x^i_α. As we have taken the lines of curvature as the coordinate curves on the shock surface we have

$$a_{12} = b_{12} = 0. \tag{37}$$

Moreover if $a^{\alpha\beta}$ denote the components of the tensor conjugate to $a_{\alpha\beta}$, namely,

$$a_{\alpha\beta}a^{\alpha\gamma} = \delta^\gamma_\beta ,$$

we obtain

$$a^{11} = a_{22}/a, \qquad a^{12} = a^{21} = 0, \qquad a^{22} = a_{11}/a. \tag{38}$$

In this notation the space components of the unit normal vector to the surface are given by the relation

$$\xi^i = \tfrac{1}{2}\epsilon^{\alpha\beta}e_{ijk}x^j_\alpha x^k_\beta . \tag{39}$$

The well-known Weingarten formulas give

$$\xi^i_{;\alpha} = -a^{\gamma\beta}b_{\alpha\gamma}x^i_\beta , \tag{40}$$

where ; denotes, as before, the surface covariant differentiation. Keeping these results and the assumption of uniform flow in front of the shock in mind, we get

$$u_{1n;\alpha} = u_{1i}\xi^i_{;\alpha} = -u_{1i}a^{\gamma\beta}b_{\alpha\gamma}x^i_\beta = v_\beta a^{\gamma\beta}b_{\alpha\gamma} , \tag{41}$$

where we have used the relations (11) and (17). Expanding this result and making use of the expressions in (35) and (38) we obtain

$$u_{1n;1} = -v_1k_1 ; \qquad u_{1n;2} = -v_2k_2 . \tag{42}$$

Similarly, the relation (40) yields

$$\xi^i_{;1} = -k_1x^i_1 ; \qquad \xi^i_{;2} = -k_2x^i_2 . \tag{43}$$

Now for the uniform flow

$$A^*_{i\alpha} = A_{i\alpha} , \qquad B^*_\alpha = B_\alpha , \qquad C^*_\alpha = C_\alpha .$$

Hence we can readily find [7].

$$\begin{aligned} A_{i\alpha} &= \left\{\frac{2(\rho_1u_{1n}^2 - \gamma p_1)\xi_i}{-(\gamma+1)\rho_1u_{1n}}\right\}_{;\alpha} \\ &= -\left\{\frac{B_\alpha\xi_i}{\rho_1u_{1n}} + \frac{[p]}{\rho_1u_{1n}^2}(-k_\alpha x^i_\alpha u_{1n} + \xi^i v_\alpha k_\alpha)\right\}, \end{aligned} \tag{44}$$

in which there is no summation on α. From (14), (30), and (44) we obtain

$$\begin{aligned} B_{11} &= \frac{A_{i1}x_1^i}{a_{11}} = \frac{[p]}{\rho_1 u_{1n}^2}\, k_1 \cdot u_{1n} \cdot \frac{x_1^i x_1^i}{a_{11}} = \delta u_n k_1 , \\ B_{12} &= B_{21} = 0, \\ B_{22} &= \delta u_n k_2 . \end{aligned} \tag{45}$$

Also from (6) and (42) we get

$$B_\alpha = \frac{\mp 4\rho_1 u_{1n} v_\alpha k_\alpha}{\gamma + 1} , \qquad \text{no summation on } \alpha,$$

yielding

$$B_{13} = -\frac{B_1 g_{11}}{\rho u_n} = \frac{4 g_{11} v_1 k_1}{\gamma + 1} , \tag{46}$$

and

$$B_{23} = -\frac{B_2 g_{22}}{\rho u_n} = \frac{4 g_{22} v_2 k_2}{\gamma + 1} , \tag{47}$$

where the relation (16) *viz.*, $\rho_1 u_{1n} = \rho_2 u_{2n}$ has been used.

Furthermore, from (44) we obtain

$$A_{i\alpha}\frac{u_i}{u_n} = -\left\{B_\alpha \cdot \frac{1}{\rho_1 u_{1n}} + \frac{\delta}{1+\delta}\left(-v_\alpha k_\alpha \cdot \frac{u_{1n}}{u_n} + v_\alpha k_\alpha\right)\right\},$$

in which there is no summation on α. From this we get

$$B_{31} = \frac{A_{i1}u_i g_{11}}{u_n} = v_1 g_{11} k_1 \left(\frac{4}{\gamma + 1} + \frac{\delta^2}{1+\delta}\right), \tag{48}$$

and

$$B_{32} = \frac{A_{i2}u_i g_{22}}{u_n} = v_2 g_{22} k_2 \left(\frac{4}{\gamma + 1} + \frac{\delta^2}{1+\delta}\right). \tag{49}$$

Lastly from the formula (32) we get the value of B_{33} as

$$\begin{aligned} B_{33} = &\left\{\delta u_n(1+\chi_1^2) - \left(\frac{8}{\gamma+1} + \frac{\delta^2}{1+\delta}\right)\chi_1^2\right\}k_1 \\ &+ \left\{\delta u_n(1+\chi_2^2) - \left(\frac{8}{\gamma+1} + \frac{\delta^2}{1+\delta}\chi_2^2\right)\right\}k_2 . \end{aligned} \tag{50}$$

The matrix $\| B_{ij} \|$ is thereby completely determined.
Correspondingly the d_i are given by the expressions

$$\begin{aligned} d_1 &= C_1 g_{11} = \frac{2 g_{11} v_1 k_1}{(\gamma+1)u_n}\{(\gamma-1)\rho - (\gamma+1)\rho_1\}, \\ d_2 &= C_2 g_{22} = \frac{2 g_{22} v_2 k_2}{(\gamma+1)u_n}\{(\gamma-1)\rho - (\gamma+1)\rho_1\}, \\ d_3 &= -\frac{\rho B_{33} M_n^2}{u_n}. \end{aligned} \tag{51}$$

From the relations (45) to (50) and the relations (51) the invariant character of B_{ij} and d_i can be easily seen.

7. Variation of velocity, density and pressure along the shock surface. The rate of change with respect to the arc length along the lines of curvature of the velocity components u_i is given by the following formula

$$u_{i,j}x_\alpha^j g_{\alpha\alpha} = B_{lm}D_{li}D_{mj}x_\alpha^j g_{\alpha\alpha} , \qquad \alpha \text{ not summed.}$$

When expanded this becomes

$$u_{i,j}\lambda_1^j = B_{l1}D_{li} , \qquad u_{i,j}\lambda_2^j = B_{l2}D_{li} . \tag{52}$$

Similarly the rate of change of density and pressure is given by

$$\rho_{,i}\lambda_\alpha^i = d_\alpha ,$$

and

$$p_{,i}\lambda_\alpha^i = -\rho u_n B_{j3}D_{ji}\lambda_\alpha^i ,$$

which, when expanded, becomes

$$p_{,i}\lambda_1^i = -\rho u_n B_{13} , \qquad p_{,i}\lambda_2^i = -\rho u_n B_{23} . \tag{53}$$

From (52) we readily find

$$\begin{aligned} qq_{,i}\lambda_1^i &= u_j u_{j,i}\lambda_1^i = B_{31} , \\ qq_{,i}\lambda_2^i &= u_j u_{j,i}\lambda_2^i = B_{32} , \end{aligned} \tag{54}$$

where $q^2 = u_i u_i$. Thus when the flow in the region 1 is uniform the expressions for these quantities are known from the results of Sec. 6. *These results moreover give the physical meaning of the invariants B_{ij} and d_i* . For example, B_{31} and B_{32} are the rates of change of half the square of the velocity magnitude along the lines of curvature as shown in (54).

Furthermore from (53) we get

$$\begin{aligned} p_{,i}p_{,i} &= \rho^2 u_n^2 B_{j3}B_{k3}D_{ji}D_{ki} \\ &= \rho^2 u_n^2\{B_{13}^2 + B_{23}^2 + (-B_{13}\chi_1 - B_{23}\chi_2 + B_{33})^2\}. \end{aligned} \tag{55}$$

Thus $p_{,i} \equiv 0$ only if $B_{13} = B_{23} = B_{33} = 0$. But from (46), (47), and (12) we get

$$B_{13} = \frac{4\tau_1 u_{1n}k_1}{\gamma + 1} ; \qquad B_{23} = \frac{4\tau_2 u_{1n}k_2}{\gamma + 1} . \tag{56}$$

Thus we find that Truesdell's [2] theorem in plane flows is carried over to spatial gas flows, *viz.*, when a uniform flow of a fluid crosses a shock, the pressure gradient cannot vanish at any point on the rear side of the shock if the shock is curved and oblique.

8. Determination of the derivatives of entropy behind the shock. The value of the entropy S as given by the relation (3) is

$$S = Jc_v \log\left(\frac{p}{\rho^\gamma}\right).$$

Thus

$$[S] = Jc_v \log\left\{\frac{(1 + \delta)p_1 + \delta\rho_1 u_{1n}^2}{p_1(1 + \delta)^{\gamma+1}}\right\}, \tag{57}$$

where we have used the shock relations (6) and (7).

In the case of uniform flow in front of the shock we derive, by differentiating both sides of (57), the result

$$S_{,i}x_\alpha^i = \left\{Jc_v \log\left(\frac{(1+\delta)p_1 + \delta\rho_1 u_{1n}^2}{p_1(1+\delta)^{\gamma+1}}\right)\right\}_{;\alpha} = E_\alpha \,, \tag{58}$$

where E_α can readily be evaluated with the help of the relations (15), (20) and (42). We also know that

$$S_{,i}u_i = 0,$$

which states that entropy is constant along the streamlines. This result and the relations (58) can be put in a compact form with the help of the matrix (26), as

$$S_{,j}C_{ji} = f_i \,, \tag{59}$$

where

$$f_1 = E_1 \,, \qquad f_2 = E_2 \,, \qquad f_3 = 0. \tag{60}$$

Inverting the relation (59) with the help of the relation (27) we obtain

$$S_{,i} = f_j D_{ij} \,. \tag{61}$$

9. Curvature of the streamlines behind the shock. The curvature K of a streamline in the flow under consideration is given by [8],

$$K\mu_i = -\left(\frac{p_{,i}}{\rho q^2} + \frac{c^2}{q}\, u_{k,k}\lambda_i\right),$$

where λ_i and μ_i are the components of the unit tangent vector and principal normal vector to the streamlines. Hence

$$K^2 = \frac{1}{q^4}\left(\frac{p_{,i}p_{,i}}{\rho^2} - \frac{c^4}{q^2}\, u_{k,k}^2\right).$$

When we make use of the results (32) and (55) we get the value of K^2 as

$$K^2 = \frac{u_n^2}{q^4}\left\{B_{13}^2 + B_{23}^2 + (-B_{13}\chi_1 - B_{23}\chi_2 + B_{33})^2 - \frac{B_{33}^2}{1+\chi_1^2+\chi_2^2}\right\}.$$

In this work we have confined ourselves to the evaluation of the first derivatives of the parameters of the flow behind the shock, while to determine the torsion of the streamlines we require as well the second derivatives [8] which have not yet been calculated.

10. The expression for the vorticity vector behind the shock. The components of vorticity behind the wave are given by

$$w^i = e^{ijk}u_{k,j} = e^{ijk}B_{lm}D_{lk}D_{mj} = (-k_1v_1g_{11}\lambda_2^i + k_2v_2g_{22}\lambda_1^i)\,\frac{\delta^2}{1+\delta}. \tag{62}$$

Now for w^i to be zero it is necessary as well as sufficient that either $\delta^2/(1+\delta)$ or

$$(-k_1v_1g_{11}\lambda_2^i + k_2v_2g_{22}\lambda_2^i)$$

vanish. In the latter case coefficients of both λ_1^i and λ_2^i must vanish because λ_1^i and λ_2^i

are perpendicular to each other. Thus in order that the flow be irrotational behind the shock surface at least one of the following conditions must be satisfied

(i) $\delta = 0$, no shock wave;

(ii) k_1 and k_2 are zero, i.e., the shock surface is a plane;

(iii) v_1 and v_2 are zero, i.e., the obliquity is zero which again leads to a plane;

(iv) $k_1 = 0 = v_2$;

(v) $k_2 = 0 = v_1$.

As (iv) and (v) lead to the same result, we will discuss, for the sake of definiteness, the case (v), *viz.*, $k_2 = 0 = v_1$. $k_2 = 0$ implies that the shock surface is a developable surface and as such its generators and their orthogonal trajectories form its two congruences of lines of curvature [5a]; while $v_1 = 0$ implies that the orthogonal trajectories to its generators are plane curves lying in the planes which are normal to the direction of the given uniform flow in front of the shock wave.

Now with the exception of cylinders and cones every developable surface is the tangent surface of some curve [5b] and the orthogonal trajectories of the tangent surface of a curve are the involute of the curve [5c]. But the necessary and sufficient condition that the involutes of a twisted curve be plane curves is that the curve be a cylindrical helix [5d]. Moreover, the planes of the involutes of a cylindrical helix are normal to the generators of the cylinder on which the helix lies [5e]. Therefore, if we take the helix to lie on the cylinder whose generators are in the direction of the flow then the tangent surface of this helix, developable helicoid, satisfies the condition (v). As far as a cone and a cylinder are concerned it can be easily seen that the right circular cone with its axis parallel to the direction of the uniform flow and any cylinder with generators parallel to the direction of the uniform flow satisfy the conditions (v). We thus get the theorem: *the only shock surfaces behind which the flow remains irrotational are, a plane, a right circular cone, a cylinder and a developable helicoid.* Among these shocks the conical shocks have been extensively studied [9]. In the case of cylindrical shock surface whose generators are parallel to the direction of the uniform flow it is obvious that there is no discontinuity in the flow because flow is merely tangential to the surface.

Another result follows if we multiply both sides of the Eq. (62) by ξ_i getting $w^i\xi_i = 0$. Hence, when a uniform flow breaks across a shock, the component of vorticity along the normal to the shock wave vanishes at every point of the surface. It is interesting to note that this property is also shared by contact discontinuities, i.e., vortex sheets, which are singular surfaces across which the tangential component of the velocity changes abruptly while the normal component of velocity is continuous and the expression for vorticity components on the sheets is given by [10]

$$w^i = e^{ijk}\xi_j[u_k],$$

$[u_k]$ being the jump in velocity across the vortex-sheet. The component of vorticity normal to the three-dimensional pseudo-stationary and unsteady shock fronts has also been found to vanish [11, 12].

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Note added in proof: The author has recently extended these results to unsteady shocks in fluids obeying an arbitrary equation of state [12].

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SOME INTEGRATED PROPERTIES OF SOLUTIONS OF THE WAVE EQUATION WITH NON-PLANAR BOUNDARIES*

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Abstract. Integrated properties of solutions of the wave equation with non-planar boundaries are found and applied to three dimensional supersonic flow problems and two dimensional diffraction problems.

For the problem of supersonic flow outside a cylindrical surface with generators parallel to the flow direction, a theorem is proved concerning the integrated properties of the linearized pressure distribution and the prescribed normal velocity on the surface. The theorem is a generalization of the integral relationships obtained previously and is useful in the evaluation of total lift and drag of wing-body combinations when the linear dimensions of the cross section of the body are not small as compared to the chord length.

For the diffraction of a pulse or a weak shock over a rectangular notch, a pressure integral theorem is obtained. Its usefulness is demonstrated in reducing the labor of obtaining the mean pressure distribution along any depth inside the notch at different instants for various width-height ratios of the notch.

Introduction. Theorems concerning certain integrated properties of the linearized pressure field due to planar source distributions in a supersonic stream have been presented by Lagerstrom and Van Dyke [1] and by Bleviss [2]. Extension of the theorems and their applications to a class of three dimensional problems involving biplanes or cruciform wing arrangements were presented by Ferri [3] and by Ferri and Clarke [4].

Ferri, Clarke and Ting [5] obtained a theorem regarding the pressure integral along the line of intersection of a forward Mach plane with a specified non-planar surface which represents a prismatic body of rectangular cross section mounted on a planar wing with supersonic edges. The pressure integral is related to the integral of the source distribution which in turn is related to the integral of the prescribed normal velocity on the top (or bottom) surface of the body and that on the wing surface. A similar relationship can be established if the normal velocity is prescribed on the side walls of the body. The wing surface together with the surface of the body above (or below) the wing represents a cylindrical surface in the form of a "single" step with generator parallel to the direction of the undisturbed supersonic flow.

By a further extension of the theorem in [5], it can be shown that a similar integral relationship is valid if the cylindrical surface is of the form of a "stairway" with a finite number of steps. As the number of steps becomes infinite and the size of each step infinitesimal, then a sequence of "stairways" is obtained whose limit can approximate the shape of any given cylindrical surface. Hence, the generalized integral relationship to be presented in this paper is conjectured. However, the argument which leads to the conjecture cannot be accepted as a proof of the validity of the relationship unless it can be shown that the limit of the sequence of the corresponding disturbance pressures or potentials exists and equals the corresponding value for the given cylindrical surface.

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Such a convergence proof is by no means simple. An attempt to verify the relationship directly by the same procedure used in the preceding investigations has not been successful owing to the difficulty of finding the proper source distribution which corresponds to the prescribed boundary condition on the cylindrical surface. The fact that the velocity potential is a solution of the wave equation is expressed implicitly in the preceding investigations through the relationship between the integral of the pressure distribution and that of the source distribution. In the present paper the integral relationship between the pressure distribution and the prescribed normal velocity is verified by observing the fact that the velocity potential obeys the wave equation. Thus, the necessity of finding the proper source distribution is avoided.

Generalization of integral relationship. As shown in Fig. 1, a cylindrical surface, $y = F(z)$, is placed in a supersonic stream with its generator parallel to the x-axis,

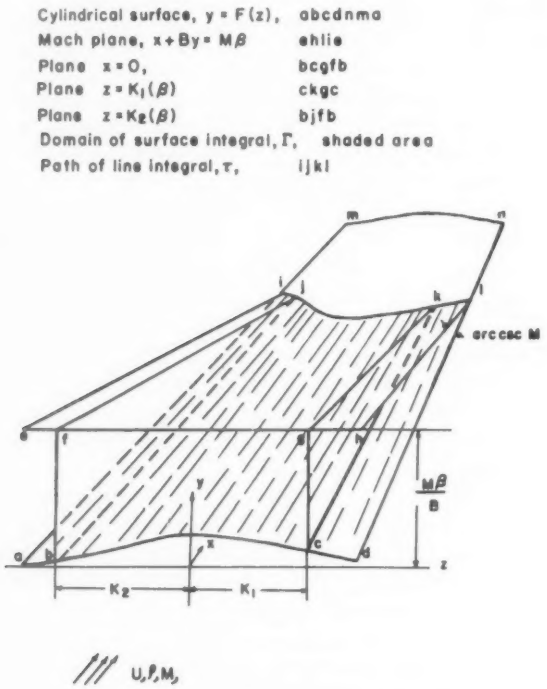
Cylindrical surface, y = F(z), abcdnma
Mach plane, x + By = Mβ ehlie
Plane x = 0, bcgfb
Plane z = K1(β) ckgc
Plane z = K2(β) bjfb
Domain of surface integral, Γ, shaded area
Path of line integral, τ, ijkl
arccsc M
Mβ/B
K2
K1
U, P, M,

FIG. 1. The generalized integral relationship.

which is the direction of the undisturbed supersonic flow with velocity U and Mach number M. $q_n[x, y = F(z), z]$ represents the prescribed small normal velocity on the cylindrical surface with

$$q_n/U \ll 1 \tag{1}$$

and

$$q_n[x, F(z), z] = 0 \quad \text{for} \quad x < 0. \tag{2}$$

It will be assumed that $q_n[x, F(z), z]$ is piece-wise continuous and that for any given Mach plane, $x + By = M\beta$, at a finite distance β from the origin, there exist two finite values $K_1(\beta)$ and $K_2(\beta)$ with $K_1 > K_2$, such that on the Mach plane, the disturbance potential φ, due to q_n, is confined inside the region $K_1 > z > K_2$, i.e.,

$$\varphi(x \leq M\beta - By, y, z) = 0$$

for

$$\infty > z \geq K_1(\beta) \quad \text{and} \quad K_2(\beta) \geq z > -\infty. \tag{3}$$

The disturbance pressure, $p(x, y, z)$, is related to the prescribed normal velocity on the cylindrical surface by the following integral relationship.

Theorem I. The integral of the y-component of the disturbance pressure force acting on the portion of the cylindrical surface, $y = F(z)$, which lies ahead of any given Mach plane parallel to the z-axis, $x + By = M\beta$ is equal to $\rho U/B$ times the integral of the prescribed normal velocity on the cylindrical surface, over the same domain of integration, i.e.

$$\iint_{\Gamma} p(x, y, z)\mathbf{n}\cdot\mathbf{j}\, dS = \frac{\rho U}{B} \iint_{\Gamma} q_n(x, y, z)\, dS, \tag{4}$$

where $\mathbf{n}$ is the unit vector normal to the cylindrical surface, $\mathbf{j}$ is the unit vector parallel to y-axis, and Γ represents the domain of integration on the cylindrical surface for

$$0 \leq x \leq M\beta - By.$$

Proof. The disturbance potential $\varphi(x, y, z)$ obeys the wave equation

$$B^2 \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} = 0. \tag{5}$$

If a vector $\mathbf{Q}(x, y, z)$ is defined as

$$\mathbf{Q} = (B\varphi_x + \varphi_y)(B\mathbf{i} - \mathbf{j}) - \varphi_z\mathbf{k} \tag{6}$$

then Eq. (5) yields

$$\operatorname{div} \mathbf{Q} = 0.$$

If the volume which is confined by the cylindrical surface $y = F(z)$, the Mach plane $x + By = M\beta$, the planes $z = K_1(\beta)$, $z = K_2(\beta)$ and $x = 0$, is designated by V and its surface by S, then Gauss' theorem [6] states

$$-\iint_{S} \mathbf{n}\cdot\mathbf{Q}\, dS = \iiint_{V} \operatorname{div} \mathbf{Q}\, dV = 0, \tag{7}$$

where n denotes the unit vector normal to the surface S and points inward. Equation (7) is valid if $\mathbf{Q}$ is continuous inside V, i.e., if φ and its first derivatives are continuous. This is certainly true when $q_n[x, F(z), z]$ is continuous. It will be shown later that Eq. (7) is valid even if q_n is piecewise continuous.

Since φ and its first derivatives vanish on planes $x = 0$, $z = K_1$ and $z = K_2$, $\mathbf{Q}$ vanishes thereon. Since the unit normal vector $\mathbf{n}_\beta$ to the Mach plane $x + By = M\beta$ is $\mathbf{n}_\beta = (\mathbf{i} + B\mathbf{j})/M$, it is clear that on the Mach plane $-\mathbf{Q}\cdot\mathbf{n} = \mathbf{Q}\cdot\mathbf{n}_\beta = 0$. Consequently

on the surface S, $\mathbf{n}\cdot\mathbf{Q}$ vanishes except the portion on the cylindrical surface, and Eq. (7) becomes

$$\iint_{\Gamma} \mathbf{n}\cdot\mathbf{Q}\, dS = 0. \tag{8}$$

The domain of integration on the cylindrical surface has been extended beyond the region of $K_2 < z < K_1$ because φ vanishes for $z \geq K_1$ or $z \geq K_2$.

For the cylindrical surface $y = F(z)$, $\mathbf{n}\cdot\mathbf{i} = 0$, and Eq. (7) becomes

$$\iint_{\Gamma} [(B\varphi_x + \varphi_y)(\mathbf{n}\cdot\mathbf{j}) + \varphi_z \mathbf{n}\cdot\mathbf{k}]\, dS = 0. \tag{9}$$

With disturbance pressure $p = -\rho U \varphi_x$ and the disturbance velocity,

$$\mathbf{q} = \varphi_x \mathbf{i} + \varphi_y \mathbf{j} + \varphi_z \mathbf{k},$$

Eq. (9) leads to

$$\begin{aligned} \iint_{\Gamma} \mathbf{j}\cdot\mathbf{n} p\, dS &= -\rho U \iint_{\Gamma} \varphi_z \mathbf{n}\cdot\mathbf{j}\, dS \\ &= \frac{\rho U}{B} \iint_{\Gamma} [\varphi_y \mathbf{n}\cdot\mathbf{j} + \varphi_z \mathbf{n}\cdot\mathbf{k}]\, dS \\ &= \frac{\rho U}{B} \iint_{\Gamma} \mathbf{q}\cdot\mathbf{n}\, dS = \frac{\rho U}{B} \iint_{\Gamma} q_n\, dS. \end{aligned} \tag{10}$$

The remainder of the proof of the theorem is to show that Eq. (7) is valid even when q_n is only piecewise continuous.

If $q_n[x, F(z), z]$ has a jump discontinuity across a curve on the cylindrical surface Γ, and the first derivatives of the disturbance potential are discontinuous across a surface S_c inside the volume V, the discontinuity surface S_c will be a characteristic surface—a Mach cone or an envelope of Mach cones.

A unit vector, $\mathbf{n}_c$, normal to the discontinuity surface, S_c , is also normal to a Mach cone and can be written in general as

$$\mathbf{n}_c = (\mathbf{i} - B\cos\lambda\mathbf{j} - B\sin\lambda\mathbf{k})/M \tag{11}$$

where λ is a parameter. Due to the discontinuity surface, a term

$$\iint_{S_c} \delta(\mathbf{Q}\cdot\mathbf{n}_c)\, dS$$

should be added to the right side of Eq. (7). Here $\delta(\ \ )$ express the jump of the quantity inside the parenthesis across the surface S_c .

From Eqs. (6) and (11), it follows:

$$\begin{aligned} \mathbf{Q}\cdot\mathbf{n}_c &= [(B\varphi_x + \varphi_y)(1 + \cos\lambda) + \varphi_z \sin\lambda]B/M \\ &= \text{grad}\ \varphi\cdot\boldsymbol{\tau}_c \\ &= |\ \boldsymbol{\tau}_c\ |\ \varphi_\tau\ , \end{aligned}$$

where

$$\tau_c = [(B\mathbf{i} + \mathbf{j})(1 + \cos\lambda) + \sin\lambda\mathbf{k}]B/M$$

and φ_τ is the derivative of φ in the direction of τ_c .

With $\tau_c \cdot n_c = 0$, or τ_c tangential to the surface S_c , φ_τ represents a derivative of φ along the surface S_c . Since φ is continuous across the surface S_c , φ_τ is also continuous, i.e., $\delta(\varphi_\tau) = 0$. Hence, the additional term to Eq. (7) vanishes,

$$\iint_{S_c} \delta(\mathbf{Q}\cdot\mathbf{n}_c)\, dS = \iint_{S_c} |\, \tau_c \,|\, \delta(\varphi_\tau)\, dS = 0$$

and Eqs. (7) to (10) are valid when q_n is piecewise continuous.* Thus concludes the proof of the theorem.

If Γ_1 and Γ_2 are the areas on the cylindrical surface $y = F(z)$ ahead of the Mach planes $x + By = \beta_1 M$ and $x + By = \beta_2 M$, respectively, the theorem or Eq. (4) yields

$$\iint_{\Gamma_1 - \Gamma_2} p\mathbf{n}\cdot\mathbf{j}\, dS = \frac{\rho U}{B} \iint_{\Gamma_1 - \Gamma_2} q_n\, dS. \tag{12}$$

When the two Mach planes are very close to each other, i.e., $M\beta_1 - M\beta_2 = dx$, the following result is obtained.

Corollary. The integral of the disturbance pressure along the curve of intersection, T, of the cylindrical surface $y = F(z)$ with the Mach plane $x + By = M\beta$ is related to the integral of the prescribed normal velocity $q_n[x, F(z), z]$ as follows

$$\int_T p \sin\sigma\mathbf{n}\cdot\mathbf{j}\, dT = \frac{\rho U}{B} \int_T q_n \sin\sigma\, dT, \tag{13}$$

where σ represents the angle between the x-axis and the tangent to the curve T.

Equation (13) is obtained readily from Eq. (12) when dS is replaced by $dx\, dT \sin\sigma$ and the area $\Gamma_1 - \Gamma_2$ becomes a strip along the curve T with thickness equal to $dx \sin\sigma$.

On the basis of linearized theory, the prescribed normal velocity q_n on the cylindrical surface $y = F(z)$ can be created by deforming the cylindrical surface slightly into a surface $y = F(z) = \epsilon E(x, y, z)$, if

$$q_n[x, F(z), z] = \epsilon U E_x[x, F(z), z](1 + F_z^2)^{-1/2} \tag{14}$$

where $\epsilon \ll 1$.

With this equation the integrals of disturbance pressure on an almost cylindrical surface in a supersonic stream are obtained from Eqs. (4) and (13).

Application to wing-body interference. In studying wing-body interference, the body is usually represented by an almost cylindrical surface with generator parallel to the direction of the supersonic stream (x-axis). The wing, which is planar with supersonic edges, coincides with the x-z plane while the x-y plane is the plane of symmetry. The surface of the body above (or below) the wing together with the upper (or lower) surface of the wing form the cylindrical surface $y = F(z)$ (Fig. 2).

*This is a sufficient condition but not a necessary condition for the validity of the theorem.

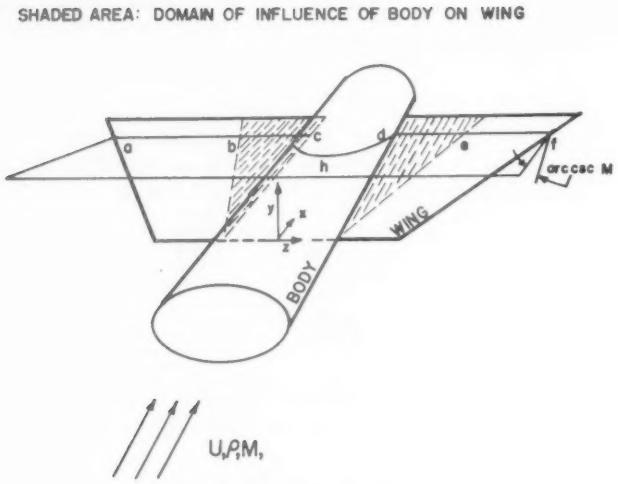

FIG. 2. Wing-body interference.

If T_B and T_W represent the line of intersection of the Mach plane, $x + By = \beta M$, with the body and the wing respectively, the corollary, Eq. (13) becomes:

$$\int_{T_B} p \sin \sigma \mathbf{n} \cdot \mathbf{j}\, dT + \int_{T_W} p\, dz = \frac{\rho U}{B} \int_{T_B} q_n \sin \sigma\, dT + U \int_{T_W} \vartheta\, dz, \qquad (15)$$

where $\vartheta(x, z)$ is the inclination of the wing surface with respect to x-axis.

This relationship can serve as a check to the numerical results of the pressure distribution obtained by analytic methods [7, 8, 9, 10].

If the wing has a straight unswept trailing edge, the total lift acting on the region Γ of the upper (or lower) surface of the wing-body combination ahead of the Mach plane passing through the trailing edge is expressed by the term on the left side of Eq. (4). This lifting force which consists of the total lift acting on the wing and the lift acting on the part of the body is equal to the integral of prescribed normal velocity represented by the right side of the same equation. Therefore, if the pressure distribution on the body is obtained by analytical methods [7, 8, 9, 10], the total lift and the center of pressure can be obtained by virtue of the theorem and the corollary, i.e., Eq. (15). The labor of calculating the pressure distribution on the wing is saved.

In the special case where the inclination of the wing surface inside the region of influence of the body is constant in the spanwise direction, the total drag can be evaluated with the help of the theorems without calculating the pressure distribution on the wing due to the interference of the body [5, 11].

For wings with subsonic edges at an angle of attack, the theorem, in general, will not be helpful in the evaluation of lift or drag. However, for the special case of a rectangular wing, the theorem can be applied to obtain expressions for the lift, the drag, and the center of pressure of the wing without calculating the detailed pressure distribution [11].

Integral relationship in diffraction problems. Since the linearized two-dimensional unsteady flow obeys the wave equation,

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{1}{C^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{16}$$

it is expected that an integral relationship similar to that for the linearized supersonic three-dimensional flow can be found. In order to demonstrate its usefulness, an integral relationship is established for a special diffraction problem—the diffraction of a weak shock or pulse over a rectangular notch.

As shown in Fig. 3, a plane weak shock ($x = Ct$), which travels parallel to the ground

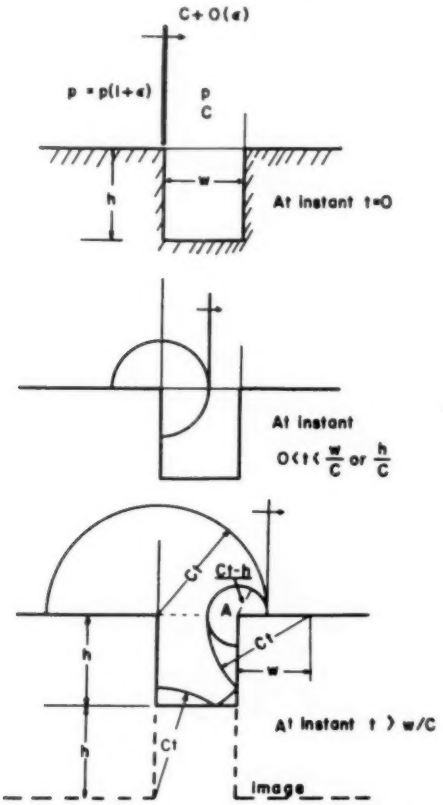

FIG. 3. Diffraction of a weak shock over a rectangular notch.

($y = 0$) is passing over a two-dimensional rectangular notch of width w and depth h. The pressure, density and speed of sound of the air at rest in front of the shock are P, ρ and C respectively. Deviations from the states at rest are defined as disturbance quantities. For example, if the notch is absent, the disturbance pressure and potential are $\epsilon PH(Ct - x)$ and $\epsilon C(1/\gamma)$. $(x - Ct)H(Ct - x)$ respectively, where ϵ is the strength of the shock and $H(Ct - x)$ is the unit function, i.e., $H = 1$ for $Ct > x$ and $H = 0$ for $Ct < x$. The disturbance due to the notch will be confined inside the forward cone $Ct = (x^2 + y^2)^{1/2}$. Across the cone, the disturbance potential and its first derivatives are continuous.

Experimental investigations of this problem in shock tubes were reported by Smith [13] and Coulter [14]. Theoretical solution to this problem can be obtained by the method outlined in [9] and [12]. Nevertheless, an integral relationship will be established and save much labor in getting numerical results.

Theorem II. Twice the value of the line integral of disturbance pressure across the notch at a depth $d(0 > y = -d > -h)$ at an instant t_0 equals the difference of the integral of disturbance pressure on the ground from the corresponding integral if the notch is absent at the "retarded" instant $t = t_0 - d/C$ plus the same at $t = t_0 - (2h - d)/C$, i.e.,

$$2\int_0^w p(x, -d, t_0)\,dx = \left[\epsilon P\int_{-Ct}^{Ct} H(Ct - x)\,dx - \int_{-Ct}^{0} p\,dx - \int_{w}^{Ct} p\,dx\right]_{\substack{y=0\\ t=t_0-d/C}} + \left[\epsilon P\int_{-Ct}^{Ct} H(Ct - x)\,dx - \int_{-Ct}^{0} p\,dx - \int_{w}^{Ct} p\,dx\right]_{\substack{y=0\\ t=t_0-(2h-d)/C}}. \tag{17}$$

*Proof.** With characteristic coordinates $\alpha = Ct - y$ and $\beta = Ct + y$, the wave equation becomes

$$4\varphi_{\alpha\beta} = \varphi_{xx}\,. \tag{18}$$

The α- and β-axes together with the x-axis form an orthogonal system. The plane $\beta = Ct_0 - d = \text{const.}$ β_0 passes through the depth $y = -d$ at the instant $t = t_0$. On

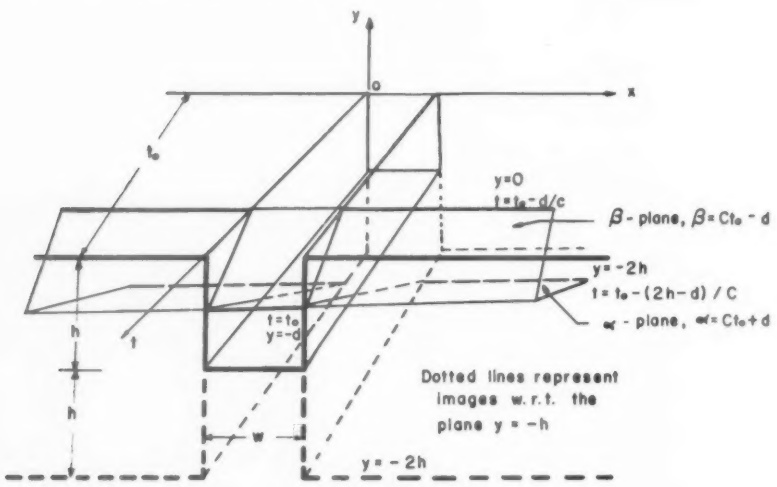

FIG. 4. Integral relationship for diffraction problems.

this plane, the cone is represented by a parabola $x^2 = \alpha\beta_0$ and the shock front is represented by a straight line $2x + \alpha = \beta_0$ which is tangential to the parabola at point $x = \beta_0$ (Fig. 4).

If S designates the area confined by the parabola, the ground $\alpha = \beta_0$, the side walls

*For the convenience of those who are interested in diffraction problems, the proof is given here without relying on the previous theorem.

of the notch $x = 0$ and $x = w$, and the line $\alpha = \beta_0 + 2d$ at the depth $y = -d$, then the divergence theorem in the plane $\beta = \beta_0$ states (p. 88, Ref. [6]):

$$\int_\Lambda (4\varphi_\beta \, dx + \varphi_x \, d_\alpha) = \int_\Lambda [4\varphi_\beta \cos(\lambda, x) - \varphi_x \cos(\lambda, \alpha)] \, d\lambda$$
$$= \int_S (4\varphi_{\alpha\beta} - \varphi_{xx}) \, dS = 0, \tag{19}$$

where Λ designates the contour of the area S. If Λ_1, denotes the part of the contour on the ground, Λ_2 on the side walls, Λ_3 along the depth $y = -d$, and Λ_4 and Λ_5 denote the portions of the parabola above and below the ground level respectively (Fig. 5) then

$$\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5 .$$

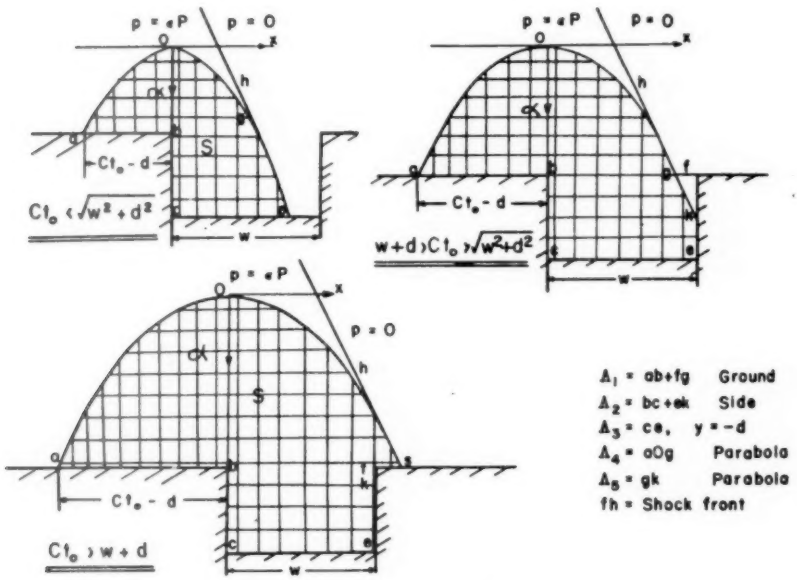

FIG. 5. β-planes, $Ct + y = Ct_0 - d$.

On the side walls, $\varphi_x = 0$ and $\cos(\lambda, x) = 0$, therefore, the line integral of Eq. (19) along Λ_2 vanishes. Across the parabola (Mach cone), φ and its first derivatives are continuous, therefore, along Λ_5, φ_β, φ_x and the corresponding line integral vanish.* Along Λ_4, $\varphi_x = \epsilon C/\gamma$ and $\varphi_\beta = -\epsilon C/(2\gamma)$ and the corresponding line integral becomes:

$$\int_{\Lambda_4} [4\varphi_\beta \, dx + \varphi_x \, dx] = 4\epsilon C(Ct_0 - d)/\gamma .$$

*When $Ct > d + w$, Λ_5 disappears from Λ, however, it does not affect the result, because the line integral along Λ_5 vanishes anyway.

Along Λ_1 , $\varphi_\nu = 0$ and $\varphi_\beta = (\varphi_\nu + \varphi_t/C)/2 = -p/(2\rho C)$, and Eq. (19) becomes

$$\int_{-(Ct_0-d)}^{0} p(x, 0, t_0 - d/C)\, dx + \int_{w}^{Ct_0-h} p(x, 0, t_0 - d/C)\, dx - \int_{-(Ct_0-d)}^{Ct_0-d} \epsilon PH(Ct_0 - d)\, dx = -\int_0^w p(x, -d, t_0)\, dx + \rho C \int_0^w \varphi_\nu(x, -d, t_0)\, dx, \tag{20}$$

the terms on the right side corresponding to the line integral along Λ_3 . For $t_0 < (d^2 + w^2)^{1/2}/C$, Λ_3 has been extended to the full width of the notch, because the integral in the additional segment vanishes.

By the method of images, the bottom of the notch plane $y = -h$, can be removed and considered as a plane of symmetry, i.e.,

$$\varphi(x, y, z) = \varphi[x, -(2h + y), z]. \tag{21}$$

Instead of the plane $\beta = Ct_0 - d$, the other characteristic plane $\alpha = Ct_0 + d$ is considered and the result corresponding to Eq. (20) is obtained,

$$\left[\int_{-Ct}^{0} p(x, -2h, t)\, dx + \int_{w}^{Ct} p(x, -2h, t) - \int_{-Ct}^{Ct} \epsilon PH(Ct - x)\, dx\right]_{t=t_0-(2h-d)/C} = -\int_0^w p(x, -d, t_0)\, dx - \rho C \int_0^w \varphi_\nu(x, -d, t_0)\, dx. \tag{22}$$

With the condition of symmetry Eq. (21), the sum of Eq. (20) and Eq. (22) is equivalent to Eq. (17). Hence, the theorem is proved.

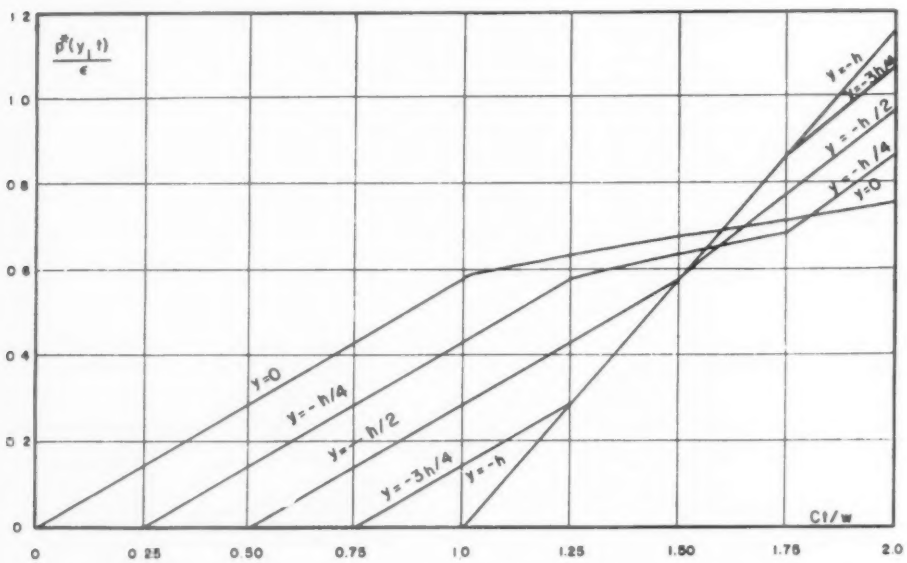

FIG. 6. Mean disturbance pressure across the width of the notch for $h/w = 1$.

The mean value of disturbance pressure across the width of the notch at depth, $y = -d$, is defined as

$$p^*(y = -d, t) = \frac{1}{w}\int_0^w p(x, -d, t)\, dx$$

and Theorem II, Eq. (17), becomes

$$p^*(y = -d, t_0) = \frac{1}{2w}\left[\epsilon P\int_{-Ct}^{Ct} H(Ct - x)\, dx - \int_{-Ct}^{0} p\, dx - \int_{w}^{Ct} p\, dx\right]_{\substack{y=0\\ t=t_0-d/C}} + \frac{1}{2w}\left[\epsilon P\int_{-Ct}^{Ct} H(Ct - x)\, dx - \int_{-Ct}^{0} p\, dx - \int_{w}^{Ct} p\, dx\right]_{\substack{y=0\\ t=t_0-(2h-d)/C}}. \tag{23}$$

With this relationship, the mean value of disturbance pressure along any depth inside the notch can be obtained from the pressure integral on the ground. The latter is obtained by the method presented in [9] and [12]. Detailed steps are given in [11]. The variations of mean pressure across the width of the notch at $y/h = 0$, $-\frac{1}{4}$, $-\frac{1}{2}$, $-\frac{3}{4}$ and -1 with respect to time are shown in Figs. 6 and 7 for notches with width-height ratios equal to 1 and $\frac{1}{2}$ respectively.

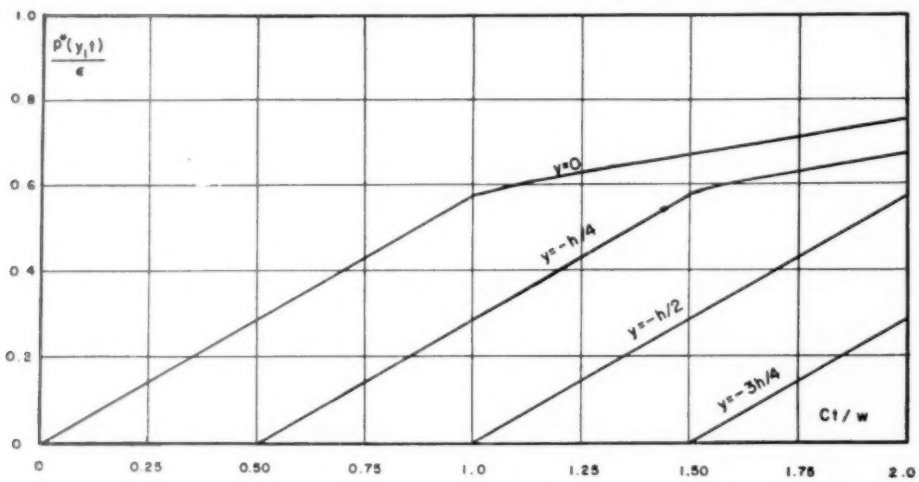

FIG. 7. Mean disturbance pressure across the width of the notch for $h/w = 2$.

Concluding remarks. For the problem of supersonic flow outside a cylindrical surface with generators parallel to the flow direction, a theorem is proved concerning the integrated properties of the linearized pressure distribution and the prescribed normal velocity on the cylindrical surface. This theorem can be extended readily to the case where there is more than one cylindrical surface present in the flow field and will be useful in the evaluation of total lift and drag of wing-body combinations and additional cylindrical bodies representing engines and (or) pylons.

For the diffraction of a pulse or a weak shock over a rectangular notch, a pressure integral theorem is obtained. By the same procedure, the theorem can be generalized for a notch of any shape, although the analytical solution of the diffraction problem is

not available. By virtue of the theorem, the time history of the mean pressure along any depth inside the notch can be obtained from the measured pressure distribution on the boundaries.

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VIBRATIONS OF TWISTED BEAMS II*

BY

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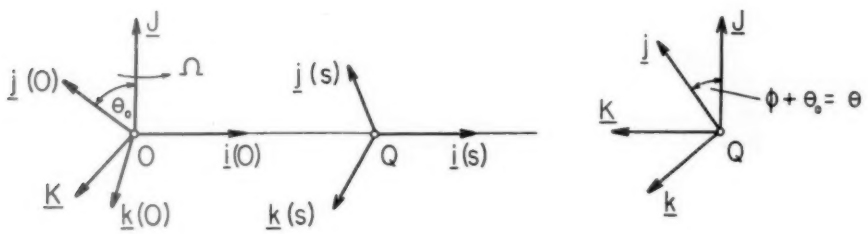
1. Introduction. The basic equations governing the transverse vibrations of a straight, twisted beam rotating about an axis, passing through one end and perpendicular to the undeflected central axis of the bar, have been discussed in an earlier paper [1].** It is the purpose of this note to explore some of the implications of these equations and to answer some unresolved questions raised in the previous work. We shall not be concerned with the problems of numerical calculations, for these have been studied extensively by others (see, for example, [2] and [3]). The results presented here are of a somewhat more qualitative nature.

The twisted beam is described in terms of a straight center line which is the locus of the centroids of the cross-sectional planes taken normal to the line. A cross-section is specified by means of the arc length s measured along the center line from a fixed origin 0 on the axis of rotation. Two triads of orthogonal unit vectors, as shown in the figure,



are used in the analysis. The first is a moving triad **i**, **j**, **k**, in which **i** is directed along the undeflected center line at a generic point Q, positive in the direction of increasing s; whereas, the vectors **j** and **k** have the directions of the principal axes of inertia of the cross section at Q. In general, the triad rotates about the **i** axis as the center line is traversed with **j**(s) at Q making an angle $\phi(s)$ with **j**(0). The second triad of unit vectors is a rotating, untwisted frame consisting of **I**, coincident with **i**; **J**, lying along the axis of rotation; and **K**, perpendicular to **I** and **J** forming a right-handed system. The vector **j**(0) forms the angle θ_0 with **J**.

If the displacement $\mathbf{u}(s, t)$ is assumed to have harmonic time dependence, $\mathbf{u}(s, t) = \mathbf{v}(s) \exp(i\lambda t)$, and $\mathbf{w}(s) = \mathbf{i} \times \mathbf{v}$, the equation of motion [see [1], Eq. (4.8)] is given by

$$\frac{d^2}{ds^2}\left\{\mathbf{B}^{-1} \cdot \frac{d^2\mathbf{w}}{ds^2}\right\} - \Omega^2 \frac{d}{ds}\left\{L(s)\frac{d\mathbf{w}}{ds}\right\} - m(s)A(s)\Omega^2\boldsymbol{\Theta}\cdot\mathbf{w} - m(s)A(s)\lambda^2\mathbf{w} = 0. \qquad (1.1)$$

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**Numbers in square brackets refer to the list of References given at the end of the paper.

The mass per unit volume and cross-sectional area are denoted by $m(s)$ and $A(s)$ respectively, while Ω is the constant angular speed of rotation. When referred to the **i**, **j**, **k** triad, the dyadic $\mathbf{B}^{-1}$ is given by

$$\mathbf{B}^{-1} = EI_1\mathbf{jj} + EI_2\mathbf{kk},$$

where $E(s)$ denotes Young's modulus, and $I_1(s)$ and $I_2(s)$ are the centroidal moments of inertia about the **j** and **k** axes respectively. The dyadic $\boldsymbol{\Theta}(s)$ is defined in terms of $\theta(s) = \phi(s) + \theta_0$ as

$$\boldsymbol{\Theta}(s) = \cos^2\theta\mathbf{jj} - \sin\theta\cos\theta(\mathbf{jk} + \mathbf{kj}) + \sin^2\theta\mathbf{kk};$$

and

$$L(s) = \int_s^l m(\xi)A(\xi)\xi\, d\xi,$$

where l is the total length of the bar.

The beam is assumed to be elastically supported at the axis of rotation. Consequently, at $s = 0$

$$\mathbf{w} = \mathbf{w}' - \boldsymbol{\varepsilon}\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' = 0, \tag{1.2}$$

where primes and Roman numerals denote differentiation with respect to s and

$$\boldsymbol{\varepsilon} = \epsilon_1\mathbf{jj} + \epsilon_2\mathbf{kk}, \qquad \epsilon_1 \geq 0, \qquad \epsilon_2 \geq 0.$$

In the **i**, **j**, **k** coordinate system, ϵ_2 is the ratio of the bending moment about the **k** axis to the angle of inclination of the beam with respect to the (**i**, **k**)-plane. A similar interpretation holds for ϵ_1 . The end $s = l$ is assumed to be free; therefore,

$$\mathbf{B}^{-1}\cdot\mathbf{w}'' = (\mathbf{B}^{-1}\cdot\mathbf{w}'')' = 0 \tag{1.3}$$

there. It is the purpose of this study to examine some of the properties of the eigenvalue problem defined by the differential equation (1.1) and boundary conditions (1.2) and (1.3).

2. Positive definiteness of the Rayleigh quotient. The Rayleigh quotient, $R(\mathbf{w}) = D(\mathbf{w})/H(\mathbf{w})$, which corresponds to the eigenvalue problem defined by Eqs. (1.1), (1.2), and (1.3), has been discussed in [1]. It was found there that

$$D(\mathbf{w}) = \int_0^l (\mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' + \Omega^2L\mathbf{w}'\cdot\mathbf{w}' - mA\Omega^2\mathbf{w}\cdot\boldsymbol{\Theta}\cdot\mathbf{w})\, ds + \mathbf{w}'(0)\cdot\boldsymbol{\varepsilon}^{-1}\cdot\mathbf{w}'(0), \tag{2.1}$$

and

$$H(\mathbf{w}) = \int_0^l mA\mathbf{w}\cdot\mathbf{w}\, ds. \tag{2.2}$$

A straightforward calculation shows that each of the terms can be written as a sum of squares, but it is not obvious that $D(\mathbf{w})$ is itself positive definite since the last term is preceded by a minus sign. It is not reasonable, on physical grounds, to expect that this functional could ever be negative, for this would imply the possibility of purely imaginary frequencies. Nevertheless, it is necessary, from the mathematical point of view, to establish positive definiteness in order to apply the usual Rayleigh-Ritz procedures and comparison theorems.

The proof depends on theorems concerning M-definite and N-definite eigenvalue problems given by Kamke [4] and Kestens [5]. These results, although actually stated for scalar equations, can be extended to vector equations of the type considered here. The major change requires the use of the integral representation of the vector solution in terms of a Green's tensor rather than a scalar Green's function. The resulting algebraic structure is the same as the standard situation and the proofs can be carried through by appropriate changes in notation.

Let us assume that $D(\mathbf{w})$ is not positive definite and there is a value of Ω^2 for which the corresponding eigenvalue λ^2 is less than zero. Since λ^2 is a continuous function of Ω^2 and $\lambda^2 > 0$ for $\Omega^2 = 0$, there must be a value of $\Omega^2 > 0$ for which $\lambda^2 = 0$.

Let ω^2 be that value of Ω^2 for which $\lambda^2 = 0$. Thus ω^2 is an eigenvalue of the problem

$$(\mathbf{B}^{-1}\cdot\mathbf{w}'')'' - \omega^2(L\mathbf{w}')' - mA\omega^2\boldsymbol{\Theta}\cdot\mathbf{w} = 0, \qquad 0 < s < l, \tag{2.3}$$

$$\begin{cases} \mathbf{w} = \mathbf{w}' - \boldsymbol{\varepsilon}\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' = 0, & s = 0, \\ \mathbf{B}^{-1}\cdot\mathbf{w}'' = (\mathbf{B}^{-1}\cdot\mathbf{w}'')' = 0, & s = l. \end{cases} \tag{2.4}$$

The Rayleigh quotient R corresponding to this problem is

$$R = \frac{\int_0^l \mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}''\,ds + \mathbf{w}'(0)\cdot\boldsymbol{\varepsilon}^{-1}\cdot\mathbf{w}'(0)}{\int_0^l (mA\mathbf{w}\cdot\boldsymbol{\Theta}\cdot\mathbf{w} - L\mathbf{w}'\cdot\mathbf{w}')\,ds}. \tag{2.5}$$

Since the numerator is positive definite and the denominator is of undetermined sign, the problem is of the type designated by Kamke [4: II, III] as M-definite. He has shown that maximum-minimum theorems of the Courant type and comparison theorems hold for this problem. An upper bound can be found for the positive eigenvalues ω^2 through these theorems. Let

$$M = \max\,(EI_1\,, EI_2), \qquad 0 < s < l;$$

then direct computation shows that

$$\mathbf{w}''\cdot\mathbf{B}^{-1}\cdot\mathbf{w}'' \leq M\mathbf{w}''\cdot\mathbf{w}''.$$

In addition, if

$$\mu = \max\, mA, \qquad \nu = \min\, mA, \qquad 0 < s < l,$$

then

$$L(s) \leq \tfrac{1}{2}\mu(l^2 - s^2).$$

Furthermore if

$$\mathbf{w} = W_1\mathbf{J} + W_2\mathbf{K},$$

then

$$\mathbf{w}\cdot\boldsymbol{\Theta}\cdot\mathbf{w} = W_1^2\,,$$

and

$$\int_0^l mA\mathbf{w}\cdot\boldsymbol{\Theta}\cdot\mathbf{w}\,ds \geq \nu\int_0^l W_1^2\,ds.$$

Also, if $\alpha = \max(\epsilon_1^{-1}, \epsilon_2^{-1})$,
then

$$\mathbf{w}' \cdot \boldsymbol{\varepsilon}^{-1} \cdot \mathbf{w}' \leq \alpha \mathbf{w}' \cdot \mathbf{w}'.$$

Let ω_n^{*2} be the nth positive eigenvalue corresponding to the Rayleigh quotient

$$R^* = \frac{\int_0^l M\mathbf{w}'' \cdot \mathbf{w}'' \, ds + \alpha \mathbf{w}'(0) \cdot \mathbf{w}'(0)}{\nu \int_0^l W_1^2 \, ds - \frac{1}{2}\mu \int_0^l (l^2 - s^2)\mathbf{w}' \cdot \mathbf{w}' \, ds}. \tag{2.6}$$

We then obtain from Kamke's comparison theorems

$$\omega_n^2 \leq \omega_n^{*2}.$$

The differential equations and natural boundary conditions which arise from the Rayleigh quotient (2.6) can be found, by elementary variational techniques, to be

$$\begin{cases} MW_1^{\mathrm{IV}} - \frac{1}{2}\mu\omega^{*2}[(l^2 - s^2)W_1']' - \omega^{*2}\nu W_1 = 0, \\ W_1(0) = \alpha W_1'(0) - MW_1''(0) = 0, \\ W_1''(l) = W_1'''(l) = 0, \end{cases} \tag{2.7}$$

and

$$\begin{cases} MW_2^{\mathrm{IV}} - \frac{1}{2}\mu\omega^{*2}[(l^2 - s^2)W_2']' = 0, \\ W_2(0) = \alpha W_2'(0) - MW_2''(0) = 0, \\ W_2''(l) = W_2'''(l) = 0. \end{cases} \tag{2.8}$$

Let us first consider the simpler of the two eigenvalue problems, (2.8). Multiplying the differential equation by W_2, integrating from 0 to l, and applying partial integration together with the boundary conditions yield

$$-\tfrac{1}{2}\mu\omega^{*2} \int_0^l (l^2 - s^2)W_2'^2 \, ds = \alpha[W_2'(0)]^2 + \int_0^l MW_2''^2 \, ds.$$

Consequently, there are no positive eigenvalues ω^{*2} for problem (2.8); and the assumed solution must be found, if it exists, from problem (2.7).

Equations (2.7) can be analyzed by observing that they represent the transverse vibrations of a uniform bar rotating about an axis at $s = 0$ with a constant angular velocity $\omega^*(\mu)^{1/2}$. The bar is elastically restrained at $s = 0$ and free at $s = l$. The circular frequency of transverse vibration corresponds to $\omega^*(\nu)^{1/2}$. According to the Southwell [6] inequality, the smallest eigenvalue $\omega^{*2}\nu$ must satisfy

$$\omega^{*2}\nu \geq K + \omega^{*2}\mu,$$

where $K > 0$ (with one possible exception to be considered later). Consequently,

$$\omega^{*2}\nu > \omega^{*2}\mu. \tag{2.9}$$

However, $0 \leq \nu \leq \mu$, contradicting (2.9) unless $\omega^{*2} < 0$. Thus there are no eigenvalues $\omega^{*2} \geq 0$ and hence no eigenvalues $\omega^2 \geq 0$. Therefore in our original problem there is

no value of $\Omega^2 \geq 0$ for which $\lambda^2 = 0$, and all of the eigenvalues of the problem defined by Eqs. (1.1), (1.2) and (1.3) are positive.

With this result we can now show that for any admissible vector $\mathbf{w}$, as defined in [1], p. 255,

$$F(\mathbf{w}) = \int_0^l mA\mathbf{w}\cdot\mathbf{\Theta}\cdot\mathbf{w}\, ds - \int_0^l L\mathbf{w}'\cdot\mathbf{w}'\, ds \leq 0, \tag{2.10}$$

and hence complete the proof that $D(\mathbf{w})$ is positive definite for any admissible vector. If (2.10) were false, there would be a vector $\mathbf{u}$ such that

$$\rho = \frac{\int_0^l \mathbf{u}''\cdot\mathbf{B}^{-1}\cdot\mathbf{u}''\, ds + \mathbf{u}'(0)\cdot\mathbf{\varepsilon}^{-1}\cdot\mathbf{u}'(0)}{\int_0^l mA\mathbf{u}\cdot\mathbf{\Theta}\cdot\mathbf{u}\, ds - \int_0^l L\mathbf{u}'\cdot\mathbf{u}'\, ds} > 0.$$

A result of Kamke's [4, III, p. 257] states that there must then be an eigenvalue ω^2 of Eqs. (2.3), (2.4), and (2.5) satisfying

$$0 \leq \omega^2 \leq \rho^2.$$

However, we have just shown that such an eigenvalue cannot exist and hence $F(\mathbf{w}) \leq 0$, proving that $D(\mathbf{w}) \geq 0$ for all admissible functions.

The only exception to the preceding argument occurs when the constant K in the Southwell inequality vanishes and $\mu = \nu$. This takes place when the bar is simply supported, $\mathbf{w}(0) = \mathbf{w}''(0) = 0$, and the maximum and minimum values of mA are equal, i.e., mA = constant. In this case,

$$\mathbf{w} = s\mathbf{J}$$

is non-trivial and yields an eigenvalue $\lambda^2 = 0$ for any value of Ω^2. Noting the relation between $\mathbf{w}$ and the displacement $\mathbf{u}$, we see that this solution corresponds to the beam's remaining straight, always lying in the plane of rotation, and making a constant angle with the $\mathbf{i}$-direction. However, the non-negative character of $D(\mathbf{w})$ is preserved if we regard this situation as the limiting case which occurs when ϵ_1^{-1}, ϵ_2^{-1} approach zero.

3. Difference equations and continuity conditions. The preceding section, together with the results contained in [1], shows that the lowest eigenvalue of the twisted, rotating beam is described by the minimum principle

$$\lambda^2 = \min_{W_1} D(\mathbf{w})/H(\mathbf{w}),$$

where the class W_1 of admissible vectors $\mathbf{w}$ consists of all vectors $\mathbf{w}$ having continuous third derivatives, piecewise continuous fourth derivatives, and satisfying $\mathbf{w}(0) = 0$. In many instances, useful comparison results could be obtained if these continuity requirements were lightened. The proof of such a minimum principle, however, would normally require all the restrictions given above. Fortunately, Courant [7] has shown, by a much more delicate analysis, that these restrictions for the admissible functions can be reduced without disturbing any of the requisite properties of the minimizing function. His proof goes much deeper than that usually given since it includes an existence theorem as well as an analysis of the convergence of a related set of difference equations. Although his paper is concerned with the standard, second-order, Sturm-Liouville problem, he states that the results will carry over for higher order, self-adjoint problems.

The extension to the present vector problem is straightforward, requiring only a few changes in small details, and none in the basic method of proof. The algebra and notation are, however, considerably more awkward.

We shall therefore give only a brief discussion of the following theorem. Let the class W of admissible vectors contain those vectors $\mathbf{w}$ which are continuous, have continuous first derivatives, piecewise continuous second derivatives, and satisfy $\mathbf{w} = 0$ at $s = 0$. Then among the vectors in W, there exists a vector, $\mathbf{w}^{(1)}$, which renders $D(\mathbf{w})$ a minimum subject to $H(\mathbf{w}) = 1$. This vector is the first eigenvector of the eigenvalue problem (1.1), (1.2) and (1.3); the first eigenvalue is $\lambda^{(1)^2} = D(\mathbf{w})^{(1)}$). If

$$H(\mathbf{w}, \mathbf{u}) = \int_0^l mA\mathbf{w}\cdot\mathbf{u}\, ds,$$

then, more generally, there exists a vector $\mathbf{w}^{(k)}$, belonging to W, which minimizes $D(\mathbf{w})$ subject to $H(\mathbf{w}) = 1$, $H(\mathbf{w}, \mathbf{w}^{(i)}) = 0$, $i = 1, \cdots, k-1$. This vector is the kth eigenvector of the system and $\lambda^{(k)^2} = D(\mathbf{w}^{(k)})$ is the kth eigenvalue.

The integrals $D(\mathbf{w})$ and $H(\mathbf{w})$ are replaced by finite sums obtained by dividing the interval $0 \leq s \leq l$ into n equal parts of length c_n. Let $\mathbf{w} = w_1\mathbf{j} + w_2\mathbf{k}$ and $\tau = d\phi/ds$. Furthermore, let $w_{1,0}, \cdots, w_{1,\nu}, \cdots, w_{1,n}$ represent the values of w_1 at the points of division and similarly for w_2. Then $w_{1,0} = w_1(0)$ and $w_{1,n} = w_1(l)$. The ordinary difference operators will be denoted by Δ, Δ^2, $\cdots$, i.e.,

$$\Delta w_{1,\nu} = \frac{1}{c_n}[w_{1,\nu+1} - w_{1,\nu}],$$

$$\Delta^2 w_{1,\nu} = \frac{1}{c_n^2}[w_{1,\nu+1} - 2w_{1,\nu} + w_{1,\nu-1}].$$

Let

$$P_\nu(\mathbf{w}) = \Delta^2 w_{1,\nu} - w_{1,\nu}\tau_\nu^2 - 2\tau_\nu \Delta w_{2,\nu} - w_{2,\nu}\Delta\tau_\nu,$$

$$Q_\nu(\mathbf{w}) = \Delta^2 w_{2,\nu} - w_{2,\nu}\tau_\nu^2 + 2\tau_\nu \Delta w_{1,\nu} + w_{1,\nu}\Delta\tau_\nu,$$

$$R_\nu(\mathbf{w}) = \Delta w_{1,\nu} - w_{2,\nu}\tau_\nu,$$

$$N_\nu(\mathbf{w}) = \Delta w_{2,\nu} - w_{1,\nu}\tau_\nu,$$

$$G_\nu(\mathbf{w}) = w_{1,\nu}^2 \cos^2\theta_\nu - 2w_{1,\nu}w_{2,\nu}\sin\theta_\nu\cos\theta_\nu + w_{2,\nu}^2\sin^2\theta_\nu,$$

$$= [M_\nu(\mathbf{w})]^2,$$

$$T(\mathbf{w}) = \epsilon_1^{-1}[R_0(\mathbf{w})]^2 + \epsilon_2^{-1}[N_0(\mathbf{w})]^2,$$

$$J_\nu(\mathbf{w}) = w_{1,\nu}^2 + w_{2,\nu}^2,$$

$$L_\nu(s) = \sum_{k=\nu}^{n} m_k A_k k c_n,$$

$$D_n(\mathbf{w}) = c_n \sum_{\nu=0}^{n-1} \{EI_{1,\nu}P_\nu^2(\mathbf{w}) + EI_{2,\nu}Q_\nu^2(\mathbf{w}) + \Omega^2 L_\nu[R_\nu^2(\mathbf{w}) + N_\nu^2(\mathbf{w})] - m_{\nu+1}A_{\nu+1}\Omega^2 G_{\nu+1}(\mathbf{w})\} + T(\mathbf{w}),$$

and

$$H_n(\mathbf{w}) = c_n \sum_{\nu=0}^{n} m_\nu A_\nu J_\nu(\mathbf{w}).$$

Our original minimum problem can now be replaced by an ordinary algebraic eigenvalue problem by asking for the vector $\mathbf{w}_\nu$ which makes $D_n(\mathbf{w})/H_n(\mathbf{w})$ a minimum. This yields, for example, a lowest eigenvalue $\lambda_n^{(1)*}$ and a corresponding eigenvector $\mathbf{w}_\nu^{(n)}$. The eigenvector $\mathbf{w}_\nu^{(n)}$ is, of course, defined only at discrete points along $0 \leq s \leq l$. However, we may extend $\mathbf{w}_\nu^{(n)}$ to a vector $\mathbf{w}^{(n)}(s)$ defined over the entire interval by means of interpolation. Unlike the second order case, parabolic rather than linear interpolation is used.

It can then be shown that, as n becomes large, a subsequence of the $\lambda_n^{(1)*}$ converges to a limit $\lambda^{(1)*}$ and a corresponding subsequence of $\mathbf{w}^{(n)}(s)$ converges to a limiting vector $\mathbf{w}(s)$. Furthermore, $\mathbf{w}(s)$ has all the requisite continuity properties for the solution of the original differential equation and is an eigenvector with $\lambda^{(1)*}$ the corresponding eigenvalue. The proof for the higher eigenvalues follows similar lines.

As noted before, this modification of Courant's proof yields an existence theorem, shows that the continuity requirements on the class of admissible vectors can be weakened, and finally proves convergence (in the sense of subsequences) for the difference equations which result from the minimization of $D_n(\mathbf{w})/H_n(\mathbf{w})$. These equations are the algebraic Euler equations corresponding to

$$\lambda_n^{(1)*} = \min D_n(\mathbf{w})/H_n(\mathbf{w}),$$

and are found to be

$$\begin{aligned}
&\Delta^2[EI_{1,\nu}P_\nu(\mathbf{w}_\nu^{(n)})] - EI_{1,\nu+1}\tau_{\nu+1}^2 P_{\nu+1}(\mathbf{w}_\nu^{(n)}) \\
&\quad - 2\Delta[EI_{2,\nu}\tau_\nu Q_\nu(\mathbf{w}_\nu^{(n)})] + EI_{2,\nu+1}\Delta\tau_{\nu+1}Q_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \Omega^2\Delta[L_\nu R_\nu(\mathbf{w}_\nu^{(n)})] \\
&\quad + \Omega^2 L_{\nu+1}\tau_{\nu+1}N_{\nu+1}(\mathbf{w}_\nu^{(n)}) - m_{\nu+1}A_{\nu+1}\Omega^2 \cos\theta_{\nu+1}M_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \lambda_n^{(1)*}m_{\nu+1}A_{\nu+1}w_{1,\nu+1}^{(n)} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\Delta^2[EI_{2,\nu}Q_\nu(\mathbf{w}_\nu^{(n)})] - EI_{2,\nu+1}\tau_{\nu+1}^2 Q_{\nu+1}(\mathbf{w}_\nu^{(n)}) \\
&\quad + 2\Delta[EI_{1,\nu}\tau_\nu P_\nu(\mathbf{w}_\nu^{(n)})] - EI_{1,\nu+1}\Delta\tau_{\nu+1}P_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \Omega^2\Delta[L_\nu N_\nu(\mathbf{w}_\nu^{(n)})] \\
&\quad - \Omega^2 L_{\nu+1}\tau_{\nu+1}R_{\nu+1}(\mathbf{w}_\nu^{(n)}) + m_{\nu+1}A_{\nu+1}\Omega^2 \sin\theta_{\nu+1}M_{\nu+1}(\mathbf{w}_\nu^{(n)}) - \lambda_n^{(1)*}m_{\nu+1}A_{\nu+1}w_{2,\nu+1}^{(n)} = 0,
\end{aligned}$$

with the natural boundary conditions

$$\begin{aligned}
P_n(\mathbf{w}_\nu^{(n)}) &= Q_n(\mathbf{w}_\nu^{(n)}) = 0, \\
\Delta[EI_{1,n}P_n(\mathbf{w}_\nu^{(n)})] &= \Delta[EI_{2,n}Q_n(\mathbf{w}_\nu^{(n)})] = 0, \\
\Delta w_{1,0}^{(n)} - \epsilon_1 EI_{1,0}P_0(\mathbf{w}_\nu^{(n)}) &= 0, \\
\Delta w_{2,0}^{(n)} - \epsilon_2 EI_{2,0}Q_0(\mathbf{w}_\nu^{(n)}) &= 0.
\end{aligned}$$

As a direct consequence of these results, we obtain Courant's maximum-minimum principle. Let $\mathbf{v}^{(i)}$, $i = 1, \cdots, n-1$, be a set of independent vectors and let W be the class of vectors $\mathbf{w}$ which have a continuous first derivative, a piecewise continuous second derivative, and satisfy

$$H(\mathbf{w}, \mathbf{v}^{(i)}) = 0, \qquad i = 1, \cdots, n-1.$$

If

$$M(\mathbf{v}^{(i)}) = glb_W D(\mathbf{w})/H(\mathbf{w}),$$

where glb_W stands for the greatest lower bound with respect to all vectors $\mathbf{w}$ in W, then

$$\lambda^{(n)2} = \max_{\mathbf{v}^{(i)}} M(\mathbf{v}^{(i)}).$$

4. Some applications of the minimum principle. Two simple applications of the comparison theorems implied by the Courant maximum-minimum principle will be mentioned here. Until now we have considered the case in which the fixed end of the beam occurs at the axis of rotation $s = 0$. Let us now assume that the beam is mounted on a finite hub of radius d so that the flexible portion is in the range $d \leq s \leq l$. Then the differential equation (1.1) holds over the range $d < s < l$ rather than $0 < s < l$ and the boundary conditions (1.2) are applied at $s = d$.

Consider two beams identical save for the fact that one has a hub radius d_1, whereas the other has a hub radius d_2 ($d_1 \leq d_2$). In the first case s has the range $d_1 < s < l$, whereas $d_2 < s < l$ in the second. We shall now establish the inequality

$$\lambda^{(n)2}(d_1) \leq \lambda^{(n)2}(d_2), \tag{4.1}$$

a result which is physically reasonable.

We shall define $D_1(\mathbf{w})$ in the same fashion as $D(\mathbf{w})$ in Eq. (2.1) except that the range of integration now runs from d_1 to l and the boundary term is evaluated at $s = d_1$. We define $H_1(\mathbf{w})$, $D_2(\mathbf{w})$ and $H_2(\mathbf{w})$ correspondingly. Furthermore, let $W(d_2)$ contain those vectors $\mathbf{w}$ which, in the interval $d_2 \leq s \leq l$, have a continuous first derivative, a piecewise continuous second derivative, and satisfy $\mathbf{w}(d_2) = 0$. We shall normalize these vectors by requiring that $H_2(\mathbf{w}) = 1$. The maximum-minimum principle thus states that if

$$M_2(\mathbf{v}^{(i)}) = glb_{W(d_2)} D_2(\mathbf{w}),$$

under the condition $H_2(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, $i = 1, \cdots, n-1$, then

$$\lambda^{(n)2}(d_2) = \max_{\mathbf{v}^{(i)}} M_2(\mathbf{v}^{(i)}).$$

Similarly, let $W(d_1)$ contain all vectors $\mathbf{w}$ in $d_1 \leq s \leq l$ which have continuous first derivatives, piecewise continuous second derivatives, and satisfy $\mathbf{w}(d_1) = 0$. If

$$M_1(\mathbf{v}^{(i)}) = glb_{W(d_1)} D_1(\mathbf{w})/H_1(\mathbf{w})$$

under the condition $H_1(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, $i = 1, \cdots, n-1$, then

$$\lambda^{(n)2}(d_1) = \max_{\mathbf{v}^{(i)}} M_1(\mathbf{v}^{(i)}).$$

Let W^* be the set of vectors $\mathbf{w}^*$ defined over $d_1 \leq s \leq l$ which are found by continuing each vector belonging to $W(d_2)$ into $d_1 \leq s \leq d_2$ under the following restrictions. When $s = d_1$, $\mathbf{w}^* = 0$; $\mathbf{w}^*$ has a continuous first derivative and a piecewise continuous second derivative in $d_1 \leq s \leq l$; and $D_1(\mathbf{w}^*) - D_2(\mathbf{w}^*) < \rho$, where ρ is an arbitrarily small number. Finally, if for a given set of vectors $\mathbf{v}^{(i)}$, $i = 1, \cdots, n-1$, a vector $\mathbf{w}$ in $W(d_2)$ satisfies $H_2(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, then the corresponding continuation $\mathbf{w}^*$ satisfies

$$\int_{d_1}^{d_2} mA\mathbf{w}^* \cdot \mathbf{v}^{(i)}\, ds = 0, \qquad i = 1, \cdots, n-1;$$

i.e., $H_1(\mathbf{w}^*, \mathbf{v}^{(i)}) = 0$. Thus, W^* is included in $W(d_1)$.

With these definitions, we see that for any set of vectors $\mathbf{v}^{(i)}$ for which $H_1(\mathbf{w}, \mathbf{v}^{(i)}) = 0$,

$$glb_{W(d_1)} \frac{D_1(\mathbf{w})}{H_1(\mathbf{w})} \leq glb_{W^*} \frac{D_1(\mathbf{w}^*)}{H_1(\mathbf{w}^*)}.$$

In addition, $H_1(\mathbf{w}^*) \geq H_2(\mathbf{w}^*) = 1$. Therefore, if $H_2(\mathbf{w}, \mathbf{v}^{(i)}) = 0$, $i = 1, \cdots, n-1$,

$$glb_{W^*} \frac{D_1(\mathbf{w}^*)}{H_1(\mathbf{w}^*)} < glb_{W^*} D_2(\mathbf{w}^*) + \rho = glb_{W(d_2)} D_2(\mathbf{w}) + \rho.$$

This result may also be written as

$$M_1(\mathbf{v}^{(i)}) < M_2(\mathbf{v}^{(i)}) + \rho.$$

Maximizing over $\mathbf{v}^{(i)}$ and noting that ρ is arbitrary, we have (4.1). Inequality (4.1) agrees with the quantitative results found for the uniform beam by Boyce [8] for zero angle of inclination to the plane of rotation and by Schilhansl [9] for a general angle of inclination.

As a second example, we shall see how some information concerning the effect of twist can be obtained simply from the maximum-minimum principle. Consider a non-rotating, twisted beam, clamped at the end $s = 0$. The governing equations are Eq. (1.1) with $\Omega = 0$, Eq. (1.2) with $\boldsymbol{\varepsilon} = 0$, and Eq. (1.3). We shall also assume that $EI_1 \geq EI_2$ for $0 \leq s \leq l$. The corresponding Rayleigh quotient R is

$$R = \int_0^l \mathbf{w}'' \cdot \mathbf{B}^{-1} \cdot \mathbf{w}'' \, ds \Big/ \int_0^l mA\mathbf{w} \cdot \mathbf{w} \, ds.$$

From the inequality $EI_1 \geq EI_2$, we see by direct computation that

$$EI_2 \mathbf{w}'' \cdot \mathbf{w}'' \leq \mathbf{w}'' \cdot \mathbf{B}^{-1} \cdot \mathbf{w}'' \leq EI_1 \mathbf{w}'' \cdot \mathbf{w}''.$$

This may also be written in terms of the unit vectors $\mathbf{J}$ and $\mathbf{K}$ as

$$EI_2(W_1''^2 + W_2''^2) \leq \mathbf{w}'' \cdot \mathbf{B}^{-1} \cdot \mathbf{w}'' \leq EI_1(W_1''^2 + W_2''^2).$$

Consequently, if $\lambda_i^{(n)^2}$, $i = 1, 2$, represents the nth eigenvalue corresponding to the Rayleigh quotient R_i defined by

$$R_i = \int_0^l EI_i(W_1''^2 + W_2''^2) \, ds \Big/ \int_0^l mA(W_1^2 + W_2^2) \, ds,$$

we have from the comparison theorem

$$\lambda_2^{(n)^2} \leq \lambda^{(n)^2} \leq \lambda_1^{(n)^2}.$$

The scalar differential equations which correspond to the Rayleigh quotients R_i are

$$\left.\begin{aligned} &(EI_1 W_k'')'' - mA\lambda_1^2 W_k = 0, \\ &W_k(0) = W_k'(0) = W_k''(l) = W_k'''(l) = 0, \end{aligned}\right\} k = 1, 2 \qquad (4.2)$$

and similarly for R_2 with the subscript 1 replaced by 2.

We note that Eqs. (4.2) represent the ordinary equations of transverse vibration of an untwisted beam of flexural rigidity EI_1 and mass per unit length mA. It is important to note, however, that the beam is capable of transverse vibration in two directions, W_1 or W_2, and hence each eigenvalue is a double one (one corresponding to $W_1 \equiv 0$

and one to $W_2 \equiv 0$). Consequently, if we think of $\lambda_1^{(n)*}$ or $\lambda_2^{(n)*}$ as the nth eigenvalue of an untwisted beam vibrating in only one transverse direction, the appropriate inequality becomes

$$\left.\begin{aligned} \lambda_2^{(k)*} \leq \lambda^{(2k)*} \leq \lambda_1^{(k)*} \\ \lambda_2^{(k+1)*} \leq \lambda^{(2k+1)*} \leq \lambda_1^{(k+1)*} \end{aligned}\right\} k = 1, 2, \cdots \tag{4.3}$$

Equations (4.3) thus give bounds for the frequencies of a twisted beam in terms of those of an untwisted beam.

5. The 1-p resonance problem. The techniques used in Sect. 2 can be applied to yield more insight into the practical problem of 1-p resonance. This question, which is of interest to the aircraft propeller designer, is to determine whether there is a frequency of rotation which coincides with the lowest natural frequency of the twisted blade. If such a frequency occurs within the operating range of the propeller, undesirable resonance phenomena can take place. It is hoped that the following remarks will shed some light on the mechanism responsible for 1-p resonance.

Lo and Renbarger [10] have set up the equations for a uniform, rotating bar whose plane of bending is inclined at an angle γ to the plane of rotation. The governing differential equation, in non-dimensional form, is

$$w^{IV} - \tfrac{1}{2}\alpha^2[(1 - x^2)w']' - (\beta^2 - \alpha^2 \sin^2 \gamma)w = 0, \qquad 0 < x < 1.$$

Here α is the non-dimensional rotational frequency and β a similarly defined frequency of vibration. Boyce [11] has shown for the clamped bar that if $\alpha^2 \leq 12.36$, there is no angle of inclination γ for which 1-p resonance will take place. On the other hand, if $\alpha^2 \geq 15.2$, there will always be a value of γ for which one can have 1-p resonance. Similar results hold for other end conditions. In other words, if the beam rotates at a speed above a fixed limit, inclination to the plane of rotation will produce 1-p resonance.

Now let us consider whether it is possible to have 1-p resonance in an untwisted beam vibrating perpendicularly to the plane of rotation. In particular, we ask whether we can find variable EI and mA distributions which will permit this type of resonance.

Equations (1.1), (1.2), and (1.3) reduce in this case to

$$\begin{cases} (EIw'')'' - \Omega^2[L(s)w']' = mA\lambda^2 w, & 0 < s < l; \\ w = w' - \epsilon EIw'' = 0, & s = 0; \\ EIw'' = (EIw'')' = 0, & s = l. \end{cases} \tag{5.1}$$

For 1-p resonance $\lambda^2 = \Omega^2$, and the differential equation becomes

$$(EIw'')'' = \Omega^2[L(s)w']' + mA\Omega^2 w$$

with the boundary conditions of Eqs. (5.1). This defines an eigenvalue problem for Ω^2, and the corresponding Rayleigh quotient is

$$\Omega^2 = \frac{\int_0^l EIw''^2\, ds + \epsilon^{-1}[w'(0)]^2}{\int_0^l mAw^2\, ds - \int_0^l L(s)w'^2\, ds}.$$

If there is a positive eigenvalue Ω^2, 1-p resonance will take place.

As in Sec. 2, let

$$M = \max_s EI, \qquad \nu = \min_s mA, \qquad \mu = \max_s mA.$$

Again, the extended comparison theorems state that the positive eigenvalues Ω^2 are bounded above by the positive eigenvalues ω^2 which correspond to the Rayleigh principle

$$\omega^2 = \frac{M\int_0^l w''^2\,ds + \epsilon^{-1}[w'(0)]^2}{\nu\int_0^l w^2\,ds - \frac{1}{2}\mu\int_0^l (l^2 - s^2)w'^2\,ds}.$$

The resulting eigenvalue problem is

$$\begin{cases} Mw^{IV} + \frac{1}{2}\omega^2\mu[(l^2 - s^2)w']' = \omega^2\nu w, \\ w(0) = \epsilon^{-1}w'(0) - Mw''(0) = w''(l) = w'''(l) = 0. \end{cases}$$

However, this system is identical with Eq. (2.7). We have previously found that there are no positive eigenvalues in this case unless the beam is simply supported and the mass distribution is uniform. It is easily seen from Eqs. (5.1) that there is always an eigenvalue $\lambda^2 = \Omega^2$ in this case and 1-p resonance takes place. Furthermore, this result remains true for the simply-supported, twisted beam with uniform mass distribution as is seen from the solution

$$\mathbf{w} = s\mathbf{K}.$$

Our analysis has thus shown that, except for the simply-supported bar of uniform mass distribution, there is no EI or mA distribution which will produce 1-p resonance. In other words, if the section is not inclined to the plane of rotation, this phenomenon cannot take place.

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BOOK REVIEWS

(*Continued from p. 334*)

The molecular theory of solutions. By I. Prigogine, with the collaboration of A. Bellemans and V. Mathot. Interscience Publishers, Inc., New York, and North-Holland Publishing Co., Amsterdam, 1957. xx + 448 pp. $13.25.

It has long been recognized that the lattice theories of liquid mixtures are inherently inadequate: even the most elaborate versions fail to allow for changes in lattice parameters with composition, and more important, neglect the effect of composition upon the free volumes of the constituent molecules. Furthermore, no lattice model will yield an equation of state for the mixture. On the other hand, empirical relations such as the Benedict-Webb-Rubin equation of state, although lacking a proper theoretical basis, have been extremely successful in predicting the properties of certain mixtures from the properties of the pure components.

Professor Prigogine and his co-workers have combined the best features of both approaches into a treatment in which the mixture obeys the same reduced equation of state as the pure components, but with composition-dependent reduced variables. The theoretical justification of this is given in terms of the cell model for liquids, the difficult calculation of the free volume being avoided through the hypothesis that the reduced free volume in the mixture is equal to that of the pure components at the same reduced pressure and temperature. Although calculations made by this method seem somewhat more cumbersome than those involved in the simpler lattice theories which are most commonly used to discuss solution thermodynamics, there is no doubt that the results obtained are in considerably better agreement with experiment.

In addition to the detailed exposition and application of his own work on mixtures, Prof. Prigogine has given an excellent review of alternative approaches, including one dimensional solutions and the concept of conformal mixtures introduced by Longuet-Higgins. Quantum effects, especially in isotopic mixtures, are also treated.

This is probably the most original and stimulating discussion of the statistical mechanics of solutions to appear within the last several years. It should be useful both to those doing research in the area and to those who wish to obtain a general acquaintance with the recent literature.

STEPHEN PRAGER

Ordinary difference-differential equations. By Edmund Pinney. University of California Press, Berkeley and Los Angeles, 1958. xii 262 pp. $5.00.

Mr. Pinney has written a detailed account of practical methods of obtaining quantitative information on the solutions of difference-differential equations. He has directed his book towards the reader interested in applications, and the care with which the methods are described, and numerous illustrations make the book a valuable reference. Although nonlinear equations are discussed in the latter part of the book, these equations are nearly linear, and the book is primarily a study of a highly specialized linear analysis applicable to linear systems with constant coefficients.

Integral transform and integral representation methods are described and illustrated in Chapter I. The Euler-Laplace transform is used in Chapter II to obtain infinite series solutions of a general system of linear difference-differential equations with constant coefficients. The solution of the characteristic equation is studied in Chapter III. Nyquist's stability criterion is properly called Cauchy's index theorem. A great number of specialized first and second order linear equations with constant coefficients are studied in detail in Chapters IV through VIII. Chapter IX is described as the "first practical theory" of nonlinear difference-differential equations. A method is given for obtaining approximate solutions of a nearly linear difference-differential equation. It is a method of successive approximation that begins with a solution of the nearby linear system. The method is illustrated in Chapters X and XI. Existence and uniqueness theorems for a class of linear integro-differential systems are given in an appendix.

J. P. LASALLE

(*Continued on p. 441*)

THE BLASIUS EQUATION WITH THREE-POINT BOUNDARY CONDITIONS*

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Abstract. The Blasius equation subject to three-point boundary conditions, describing the interaction between two parallel streams, is solved by way of a series in terms of ascending powers of the ratio $\lambda = (u_1 - u_2)/u_1$, where the u_i's are the outer streams' velocities.

The first three terms of the series are analytically expressed in terms of the repeated integrals of the complementary error function (i^n erfc η) and of the repeated integrals of the square of the successive integrals of the complementary error function ($j^m i^n$ erfc η). These functions often appear in problems leading to extended heat-conduction type of equations. A recurrence formula for $j^m i^n$ erfc η is established and formulae relating the functions i^n erfc $(-\eta)$ and $j^m j^n$ erfc $(\pm\eta)$ to available tabulated values of the functions i^n erfc (η) are derived.

The first three approximations to the Blasius function and to its first two derivatives are also presented in tabulated form with four significant figures. Test on the convergence of the series has been made by comparison with some exact solutions obtained by high speed computing machine. The comparison, extended to the physically essential quantities, shows that:

(1) The Blasius function itself is slightly less accurate than its second and first derivatives.

(2) Two terms of the series for λ up to 0.5 and three terms for λ up to 0.7 yield extremely accurate results. The errors in the first two derivatives of the Blasius functions are always contained within less than one per cent.

1. Introduction. The solution of the Blasius equation with three-point boundary conditions has *per se* a considerable academic interest. The availability of closed form solution has, however, become a practical necessity in view of the recent findings which have shown the essential and unique role played by this equation in isobaric mixing flows. It can indeed be said that all the types of plane two-dimensional interactions between two streams are governed by the Blasius equation with three-point boundary conditions.

The reducibility of the basic Prandtl equations to the Blasius equation with pertinent boundary conditions was first shown by Görtler [1]** for the laminar incompressible mixing of uniform streams and, subsequently [2], for the turbulent case also. The present author showed [3] that the same happens for the compressible laminar case. A recent investigation by the present author [4] has also brought forth some evidence of an empirical correlation existing between turbulent and laminar compressible mixing. It was found that, under the assumption of a unitary turbulent Prandtl number, the velocity profiles are considerably independent of Mach numbers and density ratio and

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**Numbers in brackets refer to the bibliography at the end of the paper.

can therefore be deduced from the solution of the Blasius equation. This statement has an even larger implication insofar as it applies whenever the density ratio can be given a parabolic dependence on the velocity ratio. Thus the field of application of the Blasius function is widened to include a large variety of interactions between streams of different gases [5]. Solutions of the Blasius equation are furthermore needed in problems of laminar and turbulent mixing of non-uniform constant vorticity streams. These problems are solved by means of a series solution in terms of the "vorticity numbers": the zeroth order terms is the Blasius function and the coefficient of the equations for the higher order terms are all functions of the Blasius function and its derivatives [6].

The present paper is mainly concerned with the solution of the Blasius equation and not with its derivation for which reference is made to the pertinent literature. The solution is obtained as a series in terms of the parameter $\lambda = (u_1 - u_2)/u_1$. The first three terms (up to λ^3) are given in explicit closed form and in tabulated forms. Owing to the complicated nature of the terms of the series, its convergence could not be formally established. The results of the present method are, however, compared with some exact solutions obtained by high speed computing machine calculations.

In the course of the mathematical treatment there often appeared successive repeated integrals of the complementary error function (symbolically indicated by i^n erfc η) and successive repeated integrals of the square of the functions i^n erfc η. As this feature is common to a large variety of physical problems which can be reduced to extended heat conduction type of equations, a summary study of those functions is presented in Appendixes A and B. Formulae necessary to compute the values of i^n erfc $(-\eta)$ in terms of the already tabulated values (up to $n = 11$) of i^n erfc η are developed. A recurrence formula is established for the functions $j^m i^n$ erfc η defined as the successive integrals of the functions $(i^n \text{ erfc } \eta)^2$. Finally relationships giving the functions $j^m i^n$ erfc η in terms of the functions i^n erfc η are derived which afford a rapid evaluation of the functions themselves.

This work is part of a program of investigation on mixing phenomena carried out at the Polytechnic Institute of Brooklyn under the supervision of Prof. Antonio Ferri.

2. Solution of the Blasius equation. The equation to be solved is

$$f''' + 2ff'' = 0 \tag{1}$$

and it is subject to the following three-point boundary conditions

$$\lim_{\zeta \to +\infty} f' = 1 \qquad \lim_{\zeta \to -\infty} f' = 1 - \lambda \qquad f(0) = 0 \qquad 0 \leq \lambda \leq 1. \tag{2}$$

Quantities related to the mixing of two streams are expressible in terms of the function $f(\zeta)$ and its derivatives as follows

$$\begin{aligned} \psi &= 2(\nu u_1 x)^{1/2} f(\zeta), \\ \zeta &= \tfrac{1}{2} y \left(\frac{\nu x}{u_1}\right)^{-1/2} + k, \\ u &= u_1 f'(\zeta), \\ v &= \left(\frac{u_1 \nu}{x}\right)^{1/2} [(\zeta - k) f' - f]. \end{aligned} \tag{3}$$

In these equations x and y are space coordinates whose origin is taken to be at the point where the interaction begins, u and v are the corresponding velocity components, ψ is the stream function defined by $u = \psi_x$; $v = -\psi_x$ and ν is the kinematic viscosity coefficient. The quantity k is an arbitrary constant whose presence follows from an interesting property of the Prandtl boundary layer equations. These equations are invariant under the transformation

$$\begin{aligned} y_1 &= y + s(x), \\ x_1 &= x, \\ u(x, y) &= u(x_1 , y_1), \\ v(x, y) &= v_1(x_1 , y_1) - u(x_1 , y_1) \frac{ds}{dx}. \end{aligned} \tag{4}$$

Asymptotic boundary conditions on the x-component of the velocity remain also unchanged while boundary conditions on the y-component are changed. The mathematical implication of this fact lies in the freedom of choosing arbitrarily the third boundary condition for the Blasius equation. The physical implication is the resulting indeterminacy of the wake orientation insofar as the transformation back into the physical plane cannot be performed unless k is known. Additional physical considerations, such as the one suggested by von Kármán that a free wake be acted upon by a zero resultant force in the y-direction, will uniquely determine this constant k and thus will fix the orientation of the wake. The solution herein presented relates to a wake whose streamline through the origin satisfies the equation $y = -2k(\nu x/u_1)^{1/2}$.

The following series solution of Eq. (1) in ascending powers of λ is sought

$$f = \sum_i \lambda^i f_i \, . \tag{5}$$

Equation (5) is substituted into Eq. (1) and the coefficients of the successive powers of λ are set equal to zero. The zeroth order approximation must satisfy the following equation

$$f_0''' + 2f_0 f_0'' = 0 \tag{6}$$

with

$$\begin{aligned} \lim_{\zeta \to +\infty} f_0' &= 1, \\ f_0(0) &= 0. \end{aligned} \tag{7}$$

The pertinent solution is $f_0 = \zeta$ and it corresponds, physically, to zero mixing. The equation for the first approximation is, by taking the zeroth order solution into account

$$f_1''' + 2\zeta f_1'' = 0 \tag{8}$$

with

$$\begin{aligned} \lim_{\zeta \to +\infty} f_1' &= 0, \qquad \lim_{\zeta \to -\infty} f_1' = -1, \\ f_1(0) &= 0. \end{aligned} \tag{9}$$

The solution of Eq. (8) is

$$f_1 = (\pi)^{-1/2} \int_0^\zeta dt \int_0^t e^{-b^2}\, db - \tfrac{1}{2}\zeta. \tag{10}$$

Finally, the ith approximation ($i \geq 2$) must satisfy the equation

$$f_i''' + 2\zeta f_i'' = f_1'' R_i(\zeta), \tag{11}$$

where the $R_i(\zeta)$ are functions, at most, of the $(i-1)$th solution and are given by

$$R_i(\zeta) = -2 \sum_{h=0}^{i-1} f_{h+1} f_{i-h}''/f_1''. \tag{12}$$

Equation (11) is subject to the boundary conditions

$$\lim_{\zeta \to \infty} f_i'(0) = 0, \qquad f_i(0) = 0, \qquad (i \geq 2) \tag{13}$$

and admits the solution

$$f_i = (\pi)^{1/2} C_{1,i} f_1 - (\pi)^{-1/2} \int_0^\zeta dt \int_t^\infty e^{-b^2}\, db \int_0^b R_i(a)\, da \tag{14}$$

with

$$C_{1,i} = -(\pi)^{-1} \int_{-\infty}^{\infty} e^{-b^2}\, db \int_0^b R_i(a)\, da. \tag{15}$$

Successive approximations to the solution f can thus be readily determined to any order.

The task is considerably simplified if Eq. (14) is expressed in terms of the complementary error function and of related functions. In this problem, indeed, as well as in several other problems leading to extended heat-conduction type of equation, the solution can be expressed rather simply in terms of the following functions

$$i^n \operatorname{erfc} \eta = \int_\eta^\infty \int_{\eta_2}^\infty \cdots \int_{\eta_n}^\infty (\operatorname{erfc} \eta_n)\, d\eta_2 \cdots d\eta_n ,$$

$$j^m i^n \operatorname{erfc} \eta = \int_\eta^\infty \int_{\eta_2}^\infty \cdots \int_{\eta_m}^\infty (i^n \operatorname{erfc} \eta_n)^2\, d\eta_2 \cdots d\eta_m ,$$

$$T_{mn}(\eta) = \int_\eta^\infty (i^m \operatorname{erfc} t) \cdot (i^n \operatorname{erfc} t)\, dt.$$

Of these functions only the first ones, usually referred to as repeated integrals of the complementary error function, have been studied. Hartree [7] has shown some of their properties and applications and, more recently, Kaye [8] has tabulated them, up to the eleventh repeated integral, for positive value of the argument.

The functions $i^n \operatorname{erfc}(-\eta)$ are considered in Appendix A wherein their expressions in terms of the functions $i^n \operatorname{erfc} \eta$ and their asymptotic behavior are presented. A summary study of the functions $j^m i^n \operatorname{erfc} \eta$ is given in Appendix B. Therein the existence of a recurrence formula is proved and expressions relating the functions $j^m i^n \operatorname{erfc}(\pm\eta)$ and $T_{mn}(\pm\eta)$ to the repeated integrals of the error function are given.

If simplified notation such as

$$i^n \operatorname{erfc} \eta \equiv i^n,$$

$$j^m i^n \operatorname{erfc} \eta \equiv j^m i^n,$$

are adopted, the first approximations to the function f can be given simple analytical expressions as follows:

First approximation—

$$f_1 = \tfrac{1}{2}[i - (\pi)^{-1/2}],$$
$$f_1' = -\tfrac{1}{2}i^{0} = -\tfrac{1}{2}\operatorname{erfc}\zeta,$$
$$f_1'' = (\pi)^{-1/2}e^{-\zeta^2}.$$

Second approximation—

$$f_2 = -(16\pi)^{-1/2}[\tfrac{1}{2} - i^{0}] + i\left[\frac{i^2}{2} - \frac{1}{4}\right] + \frac{3}{4}\left[\frac{1}{3}\frac{(2)^{1/2} - 1}{(\pi)^{1/2}} - ji\right],$$

$$f_2' = \frac{i^{0}}{4} - \frac{i^{-1}}{4(\pi)^{1/2}} - \tfrac{1}{2}i^{0}i^2 + \tfrac{1}{4}(i)^2,$$

$$f_2'' = (\pi)^{-1/2}e^{-\zeta^2}[(\pi)^{-1/2}\zeta - \tfrac{1}{2} + i^2].$$

TABLE 1

Blasius equation with three-point boundary conditions—Values of f_i

ζ	$f_1(\zeta)$	$f_1(-\zeta)$	$f_2(\zeta)$	$f_2(-\zeta)$	$f_3(\zeta)$	$f_3(-\zeta)$
0.00	.0000	.0000	.0000	.0000	.0000	.0000
0.01	−.0050	.0050	.0004	−.0006	.0002	−.0002
0.02	−.0099	.0101	.0009	−.0009	.0004	−.0004
0.04	−.0195	.0204	.0017	−.0032	.0009	−.0009
0.06	−.0290	.0310	.0025	−.0034	.0013	−.0013
0.08	−.0382	.0418	.0032	−.0041	.0017	−.0018
0.10	−.0472	.0528	.0038	−.0052	.0021	−.0023
0.12	−.0559	.0640	.0044	−.0065	.0025	−.0028
0.14	−.0645	.0755	.0050	−.0077	.0029	−.0033
0.16	−.0728	.0872	.0054	−.0090	.0033	−.0038
0.18	−.0809	.0991	.0058	−.0104	.0037	−.0043
0.20	−.0888	.1112	.0063	−.0118	.0041	−.0048
0.30	−.1250	.1750	.0078	−.0196	.0058	−.0075
0.40	−.1560	.2440	.0075	−.0295	.0074	−.0104
0.50	−.1823	.3177	.0069	−.0381	.0085	−.0136
0.60	−.2041	.3959	.0056	−.0503	.0098	−.0171
0.70	−.2220	.4779	.0037	−.0610	.0107	−.0209
0.80	−.2365	.5635	.0016	−.0705	.0113	−.0250
0.90	−.2480	.6520	−.0005	−.0778	.0018	−.0295
1.00	−.2570	.7430	−.0024	−.0867	.0120	−.0342
1.20	−.2691	.9309	−.0060	−.1017	.0122	−.0437
1.40	−.2758	1.1243	−.0085	−.1126	.0121	−.0529
1.60	−.2792	1.3208	−.0102	−.1199	.0119	−.0609
1.80	−.2809	1.5191	−.0112	−.1238	.0117	−.0674
2.00	−.2816	1.7184	−.0117	−.1268	.0116	−.0720
2.20	−.2819	1.9181	−.0119	−.1280	.0115	−.0750
2.40	−.2820	2.1189	−.0120	−.1286	.0115	−.0765
2.60	−.2821	2.3179	−.0121	−.1288	.0115	−.0771
2.80	−.2821	2.5179	−.0121	−.1289	.0115	−.0773
3.00	−.2821	2.7179	−.0121	−.1289	.0115	−.0773

Third approximation—

$$f_3' = F_3 + F_3(-\infty)f_1' ,$$

$$F_3 = -\frac{i^0}{2}\left[\frac{1}{4} + \frac{\zeta^2 - 1}{2\pi} + (i^2)^2 + \frac{3}{2}\left(\frac{1}{4\pi} - \frac{1}{16} - j^2 i\right)\right]$$
$$+ \frac{1}{4}[2(\pi)^{-1/2} i^2 - (i)^2] - \frac{i^{-1} i^2}{4(\pi)^{1/2}} + \frac{i^{-1}}{4(2\pi)^{1/2}}$$
$$- \frac{i^2}{2}\left[\frac{1}{\pi} - (i)^2 - \frac{\zeta}{\pi} + \frac{3}{2} ji\right] + \frac{1}{4}\int_\zeta^\infty [i(t)]^3 \, dt,$$

$$f_3'' = F_3(-\infty)f_1'' + (\pi)^{-1/2} e^{-\zeta^2}\left[\frac{1}{4} + \frac{\zeta^2 - 1}{2\pi} + \frac{(\pi)^{-1/2}}{2}\left(\frac{\zeta}{2} + i\right)\right.$$
$$\left. - i^2(i^2 - 1) - (\pi)^{-1/2} i^2 \zeta - \frac{3}{2} j^2 i\right].$$

TABLE 2

Blasius equation with three-point boundary conditions—Values of f_i'

ζ	$f_1'(\zeta)$	$f_1'(-\zeta)$	$f_2'(\zeta)$	$f_2'(-\zeta)$	$f_3'(\zeta)$	$f_3'(-\zeta)$
0.0	−.5000	−.5000	.0454	.0454	.0222	.0222
.01	−.4943	−.5056	.0440	.0468	.0220	.0224
.02	−.4887	−.5113	.0426	.0482	.0218	.0225
.04	−.4774	−.5225	.0398	.0510	.0215	.0229
.06	−.4662	−.5338	.0370	.0538	.0211	.0233
.08	−.4550	−.5450	.0342	.0566	.0208	.0236
.10	−.4438	−.5562	.0314	.0594	.0204	.0240
.12	−.4326	−.5674	.0287	.0621	.0201	.0244
.14	−.4215	−.5785	.0260	.0648	.0197	.0247
.16	−.4105	−.5895	.0234	.0674	.0193	.0251
.18	−.3995	−.6005	.0218	.0700	.0190	.0255
.20	−.3886	−.6113	.0183	.0725	.0186	.0259
.30	−.3357	−.6643	.0080	.0839	.0166	.0280
.40	−.2858	−.7142	−.0033	.0929	.0146	.0304
.50	−.2397	−.7602	−.0109	.0990	.0123	.0332
.60	−.1981	−.8019	−.0163	.1018	.0098	.0364
.70	−.1611	−.8389	−.0195	.1014	.0075	.0399
.80	−.1289	−.8710	−.0210	.0980	.0052	.0433
.90	−.1015	−.8984	−.0209	.0920	.0034	.0450
1.00	−.0786	−.9213	−.0197	.0841	.0018	.0475
.20	−.0448	−.9551	−.0154	.0648	.0000	.0476
.40	−.0238	−.9761	−.0105	.0451	−.0010	.0439
.60	−.0118	−0.9882	−.0064	.0285	−.0012	.0369
.80	−.0054	−0.9945	−.0035	.0123	−.0008	.0278
2.00	−.0023	−0.9977	−.0017	.0086	−.0004	.0010
2.20	−.0009	−0.9991	−.0008	.0041	−.0002	.0107
2.40	−.0003	−0.9996	−.0003	.0018	−.0001	.0045
2.60	−.0001	−0.9999	−.0001	.0007	−.0001	.0018
2.80	.0000	−1.0000	.0000	.0003	.0000	.0002
3.00	.0000	−1.0000	.0000	.0001	.0000	.0000

TABLE 3

Blasius equation with three-point boundary conditions—Values of f_i''

ζ	$f_1''(\zeta)$	$f_1''(-\zeta)$	$f_2''(\zeta)$	$f_2''(-\zeta)$	$f_3''(\zeta)$	$f_3''(-\zeta)$
.00	.5642	.5642	−.1410	−.1410	−.0176	−.0176
.01	.5641	.5641	−.1410	−.1410	−.0176	−.0176
.02	.5640	.5640	−.1409	−.1409	−.0177	−.0177
.04	.5633	.5633	−.1404	−.1404	−.0177	−.0177
.06	.5622	.5622	−.1395	−.1395	−.0178	−.0178
.08	.5606	.5606	−.1384	−.1383	−.0179	−.0179
.10	.5586	.5586	−.1370	−.1367	−.0180	−.0180
.12	.5561	.5561	−.1352	−.1348	−.0180	−.0182
.14	.5532	.5532	−.1332	−.1326	−.0182	−.0184
.16	.5499	.5499	−.1309	−.1300	−.0186	−.0188
.18	.5462	.5462	−.1283	−.1271	−.0188	−.0193
.20	.5421	.5421	−.1255	−.1239	−.0190	−.0200
.30	.5156	.5156	−.1083	−.1031	−.0201	−.0225
.40	.4808	.4808	−.0874	−.0760	−.0216	−.0255
.50	.4394	.4394	−.0650	−.0448	−.0235	−.0300
.60	.3936	.3936	−.0430	−.0121	−.0265	−.0345
.70	.3456	.3456	−.0230	.0195	−.0250	−.0350
.80	.2975	.2975	−.0061	.0478	−.0218	−.0320
.90	.2510	.2510	.0070	.0708	−.0171	−.0205
1.00	.2075	.2075	.0165	.0875	.0118	−.0100
1.20	.1337	.1337	.0246	.1011	.0072	.0100
1.40	.0795	.0795	.0233	.0927	.0031	.0260
1.60	.0436	.0436	.0176	.0722	.0015	.0435
1.80	.0221	.0221	.0114	.0491	.0028	.0480
2.00	.0103	.0103	.0065	.0297	.0018	.0440
2.20	.0045	.0045	.0033	.0160	.0007	.0350
2.40	.0018	.0018	.0015	.0078	.0004	.0220
2.60	.0006	.0006	.0006	.0035	.0002	.0105
2.80	.0002	.0002	.0002	.0014	.0001	.0050
3.00	.0001	.0001	.0001	.0005	.0001	.0008

The first three approximations to the functions $f(\zeta)$, $f'(\zeta)$ and $f''(\zeta)$ have been computed to six significant figures. Values of $f_i(\zeta)$, $f_i'(\zeta)$ and $f_i''(\zeta)$, $(i = 1, 2, 3)$ are tabulated in Tables 1 and 3 with four significant figures.

3. Accuracy of the solution. The convergence of the series given by Eq. (5) could not be formally established, owing to the complexity of the relative terms. Practical indications about the rapidity of the convergence, however, are derived by comparing the results with those of some exact solutions obtained with the D12 Differential Analyzer presently in operation at the Centro di Calcolo Elettronico of the University of Naples.[1]

The following indicative values for λ were considered: $\lambda = 0.2678$; $\lambda = 0.4796$; $\lambda = 0.5541$; $\lambda = 0.6915$. Comparison was extended to the following physically meaningful quantities:

i) $f'(0)$, (proportional to the component u of the velocity along the ξ-axis);

[1]These calculations are part of a larger program of high speed machine solution of turbulent mixing flows sponsored by the United States Air Force through the Air Force Office of Scientific Research, Air Research and Development Command, under Contract AF 18(600)-693, Project No. 17500. The cooperation of Prof. Giorgio Savastano, Associate Director of the C. C. E., is gratefully acknowledged.

ii) $f''(0)$, (proportional to the shear stress along the x-axis);
iii) $\lim_{\zeta \to +\infty} [\zeta f' - f]$, (proportional, for $k = 0$, to the y-component of the velocity at the edges of the wake).

Values obtained from the exact solutions and from the first, second and third approximations to the function $f(\zeta)$ are listed in Table 4.

TABLE 4

Comparison between exact and approximate solutions

	Exact	I Approx.	II Approx.	III Approx.
$\lambda = 0.26783$				
$f'(0)$	0.8698	0.8661	0.8693	0.8697
$f''(0)$	0.1406	0.1511	0.1410	0.1407
$\lim_{\zeta \to +\infty} [\zeta f' - f]$	0.0762	0.0755	0.0764	0.0762
$\lim_{\zeta \to -\infty} [\zeta f' - f]$	0.0865	0.0755	0.0847	0.0862
$\lambda = 0.47961$				
$f'(0)$	0.7739	0.7602	0.7706	0.7730
$f''(0)$	0.2360	0.2706	0.2382	0.2362
$\lim_{\zeta \to +\infty} [\zeta f' - f]$	0.1362	0.1353	0.1381	0.1368
$\lim_{\zeta \to +\infty} [\zeta f' - f]$	0.1765	0.1353	0.1649	0.1734
$\lambda = 0.55412$				
$f'(0)$	0.7421	0.7229	0.7368	0.7406
$f''(0)$	0.2660	0.3126	0.2693	0.2663
$\lim_{\zeta \to +\infty} [\zeta f' - f]$	0.1568	0.1563	0.1600	0.1580
$\lim_{\zeta \to +\infty} [\zeta f' - f]$	0.2154	0.1563	0.1959	0.2090
$\lambda = 0.69147$				
$f'(0)$	0.6870	0.6543	0.6760	0.6833
$f''(0)$	0.3158	0.3901	0.3227	0.3169
$\lim_{\zeta \to +\infty} [\zeta f' - f]$	0.1937	0.1951	0.2009	0.1971
$\lim_{\zeta \to -\infty} [\zeta f' - f]$	0.3003	0.1951	0.2567	0.2822

The following comments are proper:

(1) The best agreement is obtained for f'', followed, in order, by f' and f.
(2) The accuracy decreases with λ and, for a given λ, is greater for ζ large than for $|-\zeta|$ large.
(3) The y-component of the velocity at the lower edge of the wake always exhibits the maximum percental error.
(4) The first two approximations are more than satisfactory up to values of $\lambda = 0.5$. The errors in $f'(0)$ and $f''(0)$ are less than one per cent.
(5) Three terms of the series are needed for values of λ greater than 0.5. With three terms the errors are nearly always contained within less than one per cent up to λ equal to 0.7. The only exception lies again in the value of the y-component of the velocity at the lower edge of the wake. The approximate value is 6% smaller than the exact one.

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Appendix A

The error function and its repeated integrals. The error function is defined by

$$\operatorname{erf} \eta = \frac{2}{(\pi)^{1/2}} \int_0^\eta e^{-t^2}\, dt \tag{A1}$$

and the complement of the error function, erfc η, by

$$\operatorname{erfc} \eta = 1 - \operatorname{erf} \eta = \frac{2}{(\pi)^{1/2}} \int_\eta^\infty e^{-t^2}\, dt. \tag{A2}$$

The nth repeated integral of erfc η is symbolically defined as

$$i^n \operatorname{erfc} \eta = \int_\eta^\infty (i^{n-1} \operatorname{erfc} t)\, dt \qquad (n \geq 1) \tag{A3}$$

with

$$i^0 \operatorname{erfc} \eta = \operatorname{erfc} \eta. \tag{A4}$$

The functions given by Eq. (A3) are tabulated, up to $n = 11$, in Ref. [8] for positive values of the variable η.

It is of interest to extend the definition of i^n erfc η to negative values of η as follows

$$i^n \operatorname{erfc} (-\eta) = \int_{-\eta}^\infty (i^{n-1} \operatorname{erfc} t)\, dt, \tag{A5}$$

and to see whether it is possible to express them as functions of the known tabulated values of i^n erfc η.

In consistence with the notation given in Eq. (A3), the first derivative of the error function is defined by

$$-\frac{d}{d\eta} (\operatorname{erfc} \eta) = i^{-1} \operatorname{erfc} \eta = \frac{2}{(\pi)^{1/2}} e^{-\eta^2}. \tag{A6}$$

It is, apparently, an even function, so that

$$i^{-1} \operatorname{erfc} (-\eta) = i^{-1} \operatorname{erfc} \eta, \tag{A7}$$

whereas

$$i^0 \operatorname{erfc}(-\eta) = 2 - i^0 \operatorname{erfc} \eta. \tag{A8}$$

By repeated application of the following recurrence formula [7], valid for $n \geq 1$

$$2ni^n \operatorname{erfc} \eta = i^{n-2} \operatorname{erfc} \eta - 2\eta i^{n-1} \operatorname{erfc} \eta \tag{A9}$$

it is then easily verified that the following general identity holds

$$i^n \operatorname{erfc}(-\eta) = (-1)^{n+1} i^n \operatorname{erfc} \eta + 2 \sum_{h=0}^{n} \left[\frac{1 + (-1)^{n-h}}{2}\right] \frac{1}{h!} \eta^h i^{n-h} \operatorname{erfc}(0). \tag{A10}$$

In deriving Eq. (A10) it has been taken into account that Eq. (A9) yields, for $\eta = 0$

$$2ni^n \operatorname{erfc}(0) = i^{n-2} \operatorname{erfc}(0)$$

or

$$i^n \operatorname{erfc}(0) = \frac{1}{2^n(\frac{1}{2}n)!}.$$

Equation (A10) gives the required relation between the repeated integrals of the error function for negative and positive values of the independent variables. Thus, for instance, the first three integrals in the negative range of η are simply expressed by

$$\begin{aligned} i \operatorname{erfc}(-\eta) &= 2\eta + i \operatorname{erfc} \eta, \\ i^2 \operatorname{erfc}(-\eta) &= -i^2 \operatorname{erfc} \eta + \eta^2 + 2i^2 \operatorname{erfc}(0), \\ i^3 \operatorname{erfc}(-\eta) &= i^3 \operatorname{erfc} \eta + 2\eta i^2 \operatorname{erfc}(0) + \eta^3/3. \end{aligned} \tag{A11}$$

Apparently these repeated integrals will not converge as $\eta \to -\infty$. Their asymptotic behavior, for $|-\eta|$ large, is

$$i^n \operatorname{erfc}(-\eta) \sim (-1)^{n+1} \frac{2}{(\pi)^{1/2}} \frac{e^{-\eta^2}}{(2\eta)^{n+1}} + 2 \sum_{h=0}^{n} \left[\frac{1 + (-1)^{n-h}}{2}\right] \frac{\eta^h}{h!} i^{n-h} \operatorname{erfc}(0). \tag{A12}$$

Appendix B

Repeated integrals of the functions $(i^n \operatorname{erfc} \eta)^2$. Let the successive integrals of the functions $(i^n \operatorname{erfc} \eta)^2$ be symbolically indicated by

$$j^m i^n \operatorname{erfc} \eta = \int_\eta^\infty (j^{m-1} i^n \operatorname{erfc} t)\, dt \qquad \begin{matrix} (m \geq 0) \\ (n \geq 0) \end{matrix} \tag{B1}$$

with

$$j^0 i^n \operatorname{erfc} \eta = (i^n \operatorname{erfc} \eta)^2. \tag{B2}$$

It is desired to express these functions in terms of the repeated integrals of the error function.

The relationship is immediate for the two particular cases: $n = -1$ (m any positive integers) and $m = 1$ ($n \geq 0$).

Indeed when $m = 1$ and $n = -1$, it is by definition

$$ji^{-1} \operatorname{erfc} \eta = \int_\eta^\infty \left[\frac{2}{(\pi)^{1/2}} e^{-t^2}\right]^2 dt, \tag{B3}$$

so that [see Eq. (A3)]

$$ji^{-1} \text{ erfc } \eta = \left(\frac{2}{\pi}\right)^{1/2} \text{erfc } [\eta(2)^{1/2}]. \tag{B4}$$

Repeated integrations easily yield the required relation between $j^m i^{-1}$ erfc η and the repeated integrals of erfc η as

$$j^m i^{-1} \text{ erfc } \eta = \frac{[2]^{1-(m/2)}}{(\pi)^{1/2}} i^{m-1} \text{ erfc } [\eta(2)^{1/2}]. \tag{B5}$$

A corresponding expression, valid for $m = 1$ and any $n \geq 0$ can be obtained by repeated integrations by parts. As it is easy to verify, the following identity will result

$$\begin{aligned} ji^n \text{ erfc } \eta = \sum_{h=1}^{n+1} (-1)^{h+1} \frac{[2(n-h+1)]!n!2^{h-1}}{(2n+1)!(n-h+1)!} \{i^{n-h} \text{ erfc } \eta \cdot i^{n-h+1} \text{ erfc } \eta \\ - \eta(i^{n-h+1} \text{ erfc } \eta)^2\} + (-1)^{n+1}(\pi)^{-1/2} \frac{n!2^{1/2+n}}{(2n+1)!} \text{erfc } [(2)^{1/2}\eta]. \end{aligned} \tag{B6}$$

In deriving Eq. (B6), the following identity

$$j^{-1}i^n \text{ erfc } \eta = -\frac{d}{d\eta}(i^n \text{ erfc } \eta)^2 = 2i^n \text{ erfc } \eta \cdot i^{n-1} \text{ erfc } \eta \tag{B7}$$

which constitutes an obvious extension of the definition (B1) to the case $m = -1$, has been taken into account.

In the most general case use must be made of a recurrence formula. This formula can be derived by successive and repeated integration by parts of Eq. (B1). By taking Eq. (A9) into consideration one obtains

$$(2n+m)j^m i^n \text{ erfc } \eta = \tfrac{1}{2}j^{m-2}i^n \text{ erfc } \eta - \eta j^{m-1}i^n \text{ erfc } \eta - j^m i^{n-1} \text{ erfc } \eta \tag{B8}$$

valid for $m \geq 1$ and $n \geq 0$.

The recurrence formula (B8) together with Eqs. (B5) and (B6) afford a rapid computation of the functions $j^m i^n$ erfc η. Their extensions to negative values of the argument are readily accomplished by means of Eq. (A10).

The first few functions $j^m i^n$ erfc η, are explicited and their values at $\eta = 0$ are given. A simplified notation such as

$$\begin{aligned} i^n \text{ erfc } \eta &\equiv i^n \\ j^m i^n \text{ erfc } \eta &= j^m i^n \end{aligned} \tag{B9}$$

is adopted. Thus

$$\begin{aligned} ji^0 &= i^0 i^{-1} - \eta(i^0)^2 - \left(\frac{2}{\pi}\right)^{1/2} i^0[\eta(2)^{1/2}], \\ j^2 i^0 &= \tfrac{1}{4}(i^0)^2 - \tfrac{1}{2}\eta j i^0 - (2)^{-3/2} i[\eta(2)^{1/2}], \\ ji &= \tfrac{1}{3}ii^0 - \tfrac{1}{3}\eta(i)^2 - \tfrac{1}{3}ji^0, \\ j^2 i &= \tfrac{1}{8}(i)^2 - \tfrac{1}{4}\eta ji - \tfrac{1}{4}j^2 i^0, \end{aligned} \tag{B10}$$

and

$$ji^0(0) = (\pi)^{-1/2}[2 - (2)^{1/2}], \qquad 4j^2i^0(0) = 1 - \frac{2}{\pi},$$
$$3ji(0) = (\pi)^{-1/2}[(2)^{1/2} - 1], \qquad 4j^2i(0) = \frac{1}{\pi} - \frac{1}{4}. \tag{B11}$$

To conclude, the integral

$$T_{mn}(\eta) = \int_\eta^\infty i^m \operatorname{erfc} t \cdot i^n \operatorname{erfc} t \, dt \qquad (n > m)$$

will be evaluated.

Repeated integrations by parts and consideration of Eq. (B6) yield

$$T_{mn}(\eta) = \frac{(-1)^h}{2}[(i^{n-h} \operatorname{erfc} \eta)^2] + \sum_{\nu=1}^{h} (-1)^{\nu-1}[i^{m+\nu} \operatorname{erfc} \eta \cdot i^{n-\nu+1} \operatorname{erfc} \eta] \tag{B12}$$

when $n - m = 2h + 1$, and

$$T_{mn}(\eta) = (-1)^h ji^{(m+n)/2} \operatorname{erfc} \eta + \sum_{\nu=1}^{h} (-1)^{\nu-1}[i^{m+\nu} \operatorname{erfc} \eta \cdot i^{n-\nu+1} \operatorname{erfc} \eta] \tag{B13}$$

when $n - m = 2h$.

—NOTES—

EFFECTS OF GRAVITATIONAL OR ELECTROMAGNETIC FIELDS ON FLUID MOTION*

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Summary. It is shown in this paper that the effect of gravity on a stratified fluid is to inhibit steady motion in the direction of gravity; that, for small values of the magnetic viscosity, the effect of a main magnetic field is to make steady weak motions of a fluid independent of the distance along the lines of force; and that, again for small values of the magnetic viscosity, the effect of a uniform electric field is to make steady weak motions of a fluid rotationally symmetric with respect to an axis in the direction of the field. These results, together with a similar one of Proudman (1916) [1] for a fluid with general rotation, enable one to state that the effects of rotation, gravity, and electromagnetic fields (for small magnetic viscosity) are to endow the fluid with a certain anisotropic rigidity by "stiffening" it along the vorticity lines, the isopycnic surfaces or lines, or the lines of force, as the case may be. In the case of a weak steady motion or relative motion, this "stiffening" has the effect of reducing the *a priori* number of physical or at least mathematical dimensions of the motion by one.

1. Introduction. Since Proudman [1] showed that the effect of rotation on weak steady relative motion of an inviscid fluid is to make it two-dimensional, in the sense that the motion is independent of the distance along the axis of rotation, experiments performed by Taylor [2], Long [3], and others have largely supported his assertion. Recently, Gariél's experiments with a stratified fluid and Lehnert's with mercury have indicated similar effects of gravity and of a magnetic field, respectively. The purpose of this paper is to describe these effects, to predict the effect of an electric current on weak steady motions of a fluid, to explain all these effects mathematically and, as far as possible, to bring out their similarity to one another. The fluid is assumed to be inviscid and incompressible throughout and, in the case of an electromagnetic field, to have negligible magnetic viscosity.

2. Effect of rotation. For convenience of comparison and for completeness, Proudman's demonstration [1] of the effect of rotation will be briefly presented. If the general rotation of the fluid is Ω, the axis of rotation can be chosen to be the x-axis. Two other axes, both rotating with angular velocity Ω, can be chosen to provide a rotating cartesian frame of reference, with coordinates (x, y, z). The velocity components relative to this rotating frame in the directions of x, y and z will be denoted by u, v, and w respectively. The equations of motion are then (Morgan [4]):

$$\frac{Du}{Dt} = -\frac{\partial P}{\partial x}, \tag{1}$$

$$\frac{Dv}{Dt} - 2\Omega w = -\frac{\partial P}{\partial y}, \tag{2}$$

$$\frac{Dw}{Dt} + 2\Omega v = -\frac{\partial P}{\partial z}, \tag{3}$$

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in which

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

and

$$P = \frac{p}{\rho} - \frac{1}{2}\Omega^2(y^2 + z^2) + \phi \tag{4}$$

with p, ρ, and ϕ as the pressure, density, and the body-force potential, respectively. The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{5}$$

For steady and weak relative motion, Eqs. (1) to (3) become

$$\frac{\partial P}{\partial x} = 0, \tag{6}$$

$$\frac{\partial P}{\partial y} = 2\Omega w, \tag{7}$$

$$\frac{\partial P}{\partial z} = -2\Omega v. \tag{8}$$

By cross-differentiation of Eqs. (7) and (8), it can be readily shown that

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{9}$$

It follows from Eq. (5) that

$$\frac{\partial u}{\partial x} = 0. \tag{10}$$

Furthermore, by differentiation of Eqs. (7) and (8) with respect to x it can readily be seen with the aid of Eq. (6) that

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial w}{\partial x} = 0. \tag{11}$$

Equations (10) and (11) indicate that the motion is independent of x. Hence, the steady and weak relative motion is two-dimensional. In a loose way, one might say that the general rotation has endowed the fluid with a certain rigidity along the vorticity lines.

It can be readily shown that for symmetric motion the radial velocity component vanishes (so that the streamlines are spirals wound around circular cylinders). This situation can be directly compared with the vanishing of the vertical velocity component in the steady weak motion of a stratified fluid to be treated in the following section, for in the case of rotation one may imagine a virtual gravity acting in the radial direction, according to Mach and Einstein.

3. Effect of gravity. One now turns to the effect of gravity on steady and weak motions of a stratified fluid. Let the z-axis be vertical. The equations of motion are

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \tag{12}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \tag{13}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho, \tag{14}$$

in which D/Dt, as usual, stands for the substantial differentiation. The equation of incompressibility is

$$\frac{D\rho}{Dt} = 0 \tag{15}$$

which permits the equation of continuity to be given by Eq. (5).

The density can be expressed as the sum of a prevailing density ρ_1, which depends only on z, and a perturbation density ρ':

$$\rho = \rho_1(z) + \rho'. \tag{16}$$

For steady weak motion, Eqs. (12) and (13) become

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0 \tag{17}$$

and hence, from Eqs. (14) and (16) one obtains

$$\frac{\partial \rho'}{\partial x} = 0, \qquad \frac{\partial \rho'}{\partial y} = 0. \tag{18}$$

Equations (17) and (18) are correct to the first order. But from Eqs. (15) and (16) it follows that

$$u \frac{\partial \rho'}{\partial x} + v \frac{\partial \rho'}{\partial y} + w \frac{d\rho_1}{dz} + w \frac{\partial \rho'}{\partial z} = 0 \tag{19}$$

so that, from Eqs. (18),

$$w = 0 \tag{20}$$

to the second order. Thus gravity inhibits steady velocity in the vertical direction, and if the disturbance is not strong enough, steady motion can occur in horizontal planes only. This situation is directly comparable to that for symmetric relative motion with general rotation, provided that that motion is steady and weak, as mentioned in the last section. In the present case, it is the isopycnic surfaces or lines that are "stiffened."

Another result comparable to Proudman's can also be obtained. If two-dimensional motion is considered to start with, so that v is zero, Eqs. (5) and (20) yield

$$\frac{\partial u}{\partial x} = 0. \tag{21}$$

Thus the motion will be one-dimensional, in the sense that the state of affairs does not change with x—in much the same way that Proudman's motion is independent of x. In both cases x is measured in a direction normal to that of gravitation or virtual gravi-

tation created by rotation. The essential one-dimensional character of steady weak motions of a stratified fluid has been experimentally demonstrated by Gariél [5].

4. Effect of a magnetic field. One now considers a uniform magnetic field $\mathbf{H}_0$ acting in the x-direction, and imagines a weak steady motion to take place in this field. The fluid is assumed to be homogeneous. The perturbation magnetic field will be denoted by $\mathbf{h}$, so that the total magnetic field is

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}. \tag{22}$$

The vector equation of motion is (all electromagnetic variables being measured in emu), with $\mathbf{g}$ as the body force per unit mass,

$$\rho \frac{D\mathbf{v}}{Dt} = -\text{grad } p + \rho\mathbf{g} + \mu\mathbf{j} \times \mathbf{H} \tag{23}$$

in which $\mathbf{v}$ is the velocity vector, μ is the magnetic permeability and $\mathbf{j}$ the current density. Since

$$\text{curl } \mathbf{H} = 4\pi\mathbf{j}, \tag{24}$$

Eq. (23) may be written

$$\rho \frac{D\mathbf{v}}{Dt} = -\text{grad } p' - \text{grad}\left(\frac{\mu H^2}{8\pi}\right) + \text{div}\left(\frac{\mu\mathbf{HH}}{4\pi}\right) \tag{25}$$

in which

$$p' = p + \rho\phi \tag{26}$$

with ϕ again denoting the potential of the body force. The intensity $\mathbf{E}$ of the electric field is related to the magnetic field by

$$\text{curl } \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}. \tag{27}$$

Furthermore, the current density, electric field, magnetic field, and velocity field are related by the equation

$$\mathbf{j} = \sigma(\mathbf{E} + \mu\mathbf{v} \times \mathbf{H}) \tag{28}$$

in which σ is the electric conductivity. The fluid being homogeneous, the equation of continuity is again given by Eq. (5).

Since the motion under consideration is weak and steady, Eq. (25) becomes, after linearization:

$$\frac{\partial p'}{\partial x} = 0, \tag{29}$$

$$\frac{\partial p'}{\partial y} = \frac{H_0}{4\pi}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right), \tag{30}$$

$$\frac{\partial p'}{\partial z} = \frac{H_0}{4\pi}\left(\frac{\partial h_z}{\partial x} - \frac{\partial h_x}{\partial z}\right). \tag{31}$$

From Eqs. (29) to (31) one obtains by cross differentiation

$$\frac{\partial}{\partial x} \text{curl } \mathbf{h} = 0 \tag{32}$$

or, by virtue of Eq. (24),

$$\frac{\partial}{\partial x}\mathbf{j} = 0 \tag{33}$$

so that the current density is independent of x. Although Eq. (33) is not needed to prove the independence of the fluid flow on x, it contains a result which is worthy of note.

The linearized form of Eq. (28) is

$$j_x = \sigma E_x , \tag{34}$$

$$j_y = \sigma(E_y + \mu H_0 w), \tag{35}$$

$$j_z = \sigma(E_z - \mu H_0 v). \tag{36}$$

Since the motion under consideration is steady, Eq. (27) becomes

$$\text{curl } \mathbf{E} = 0. \tag{37}$$

From Eqs. (24) and (34) to (37) one obtains, by the aid of Eq. (5),

$$\frac{\partial}{\partial x}\mathbf{v} = \frac{\eta}{H_0}\nabla^2\mathbf{h} \tag{38}$$

in which

$$\eta = (4\pi\mu\sigma)^{-1} \tag{39}$$

is the magnetic viscosity. If the magnetic viscosity is negligible, Eq. (38) becomes

$$\frac{\partial \mathbf{v}}{\partial x} = 0 \tag{40}$$

so that the motion does not change with x. Thus the magnetic field seems to have endowed the fluid with a certain rigidity, as if the material along the lines of forces had been stiffened to give the fluid a fibrous structure. If the motion is axisymmetric (e.g., if a sphere is towed through the fluid), from the equation of continuity one can easily show that the radial velocity is zero. The motion is then one dimensional in any meridianal plane or, if no revolving motion exists, even truly one-dimensional. It may be mentioned that for small values of the magnetic viscosity the lines of force always move with the fluid, or, to use Alfven's expression, are "frozen" into the material (Cowling [6], pp. 5–6). In the case of steady weak motions, the word "frozen" used by Alfven acquires a much stronger meaning, as explained above.

5. Effect of an electric current. Consider now a uniform electric current $\mathbf{j}_0$ flowing in the z-direction, with (r, θ, z) as the cylindrical coordinates. From Eq. (24) one obtains

$$H_\theta = 4\pi j_0 r \tag{41}$$

the other two components of the main magnetic field being zero. Eliminating $\mathbf{j}$ and $\mathbf{E}$ from Eqs. (24), (27), and (28), one obtains

$$\frac{\partial \mathbf{H}}{\partial t} = \text{curl } (\mathbf{v} \times \mathbf{H}) + \eta\nabla^2\mathbf{H} \tag{42}$$

which, for steady motion and negligible magnetic viscosity, becomes

$$\text{curl } (\mathbf{v} \times \mathbf{H}) = 0. \tag{43}$$

To the first order, one has, with (u, v, w) for (v_r, v_θ, v_z),

$$\mathbf{v} \times \mathbf{H} = (-H_\theta w, 0, H_\theta u)$$

and

$$\text{curl}\,(\mathbf{v} \times \mathbf{H}) = \left\{\frac{1}{r}\frac{\partial}{\partial\theta}(H_\theta u), -\left[\frac{\partial}{\partial r}(H_\theta u) + \frac{\partial}{\partial z}(H_\theta w)\right], \frac{1}{r}\frac{\partial}{\partial\theta}(H_\theta w)\right\}. \tag{44}$$

From Eqs. (41), (43) and (44) it follows that

$$\frac{\partial u}{\partial\theta} = 0, \tag{45}$$

$$\frac{\partial(ru)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0, \tag{46}$$

$$\frac{\partial w}{\partial\theta} = 0. \tag{47}$$

But the equation of continuity is

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial\theta} + \frac{\partial(rw)}{\partial z} = 0.$$

Thus Eq. (46) yields

$$\frac{\partial v}{\partial\theta} = 0 \tag{48}$$

which, together with Eqs. (45) and (47), states that the motion is independent of θ. The meaning of this statement can be fully realized physically by imagining an object to be towed slowly and steadily along a circular path about an axis parallel to the main electric current. According to the result just obtained, the fluid will be pushed along circular paths or along the lines of force of the main magnetic field induced by the main electric current. What has been said in the previous section about the "stiffening" of material along lines of force can be said in the present case also.

6. Concluding remarks. From the foregoing it may be concluded that, under the assumptions stated, the effects of rotation, gravity, or an electromagnetic field are to stiffen the fluid along the vorticity lines, isopycnic surfaces or lines, or lines of force, respectively. In the case of weak steady motions or relative motions, this "stiffening" has the effect of reducing the *a priori* number of physical or at least the mathematical dimensions of motion by one, in the various senses stated. In steady weak motions of a stratified fluid the velocity in the direction of gravity vanishes, and simple cases exist in which motion normal to vorticity lines or lines of forces are completely inhibited. For the motion of a conductive fluid due to the disturbance of an object slowly and steadily towed through it along a circular path, a torus-shaped mass of fluid will revolve around the axis of symmetry if a main uniform electric current is flowing longitudinally.

As already mentioned, the concept of a virtual gravitational field created by rotation helps to unify the understanding of many phenomena of fluid motion involving gravity and rotation.

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A GENERALIZATION OF LATTA'S METHOD FOR THE SOLUTION OF INTEGRAL EQUATIONS*

BY MARVIN SHINBROT (*Lockheed Aircraft Corporation*)

1. Introduction. The integral equation

$$\varphi(x) = \int_a^b k(x - t)f(t)\,dt, \qquad a < x < b \tag{1.1}$$

with a and b finite arises in many applications. As examples, we mention the problem of optimization of filters having a finite memory [1] and the problem of diffraction through a slit [2, 3]. In the first of these problems, the kernel represents the autocorrelation of the inputs to the filter; in the second, $k(x) = H_0^{(1)}(|\,x\,|)$, the first Hankel function.

Its importance notwithstanding, no general solution of (1.1) is known; with but two exceptions, only special methods exist for application to special kernels. The first exception is the method developed not long ago by Latta [4, 5]. The second—which is now subsumed under Latta's method—applies to the conceptually trivial case when the kernel has a rational Fourier transform [1].

Latta's method can be applied in the circumstance that $k(x)$ satisfies a linear differential equation (of any order) with linear coefficients. This restriction is still very far from being moderate, of course, and since no general solution of (1.1) appears to be forthcoming, it appears worthwhile to see whether there are other integral equations having the form of (1.1) which can be reduced to one of Latta's type.

In a recent paper [6], Pearson considered (1.1) with

$$k(x) = p(x)\,\log\,|\,x\,| + q(x),$$

where p and q were polynomials. Although Pearson's method has no apparent connection with Latta's, the fact that $\log\,|\,x\,|$ is a Latta kernel leads one to consider kernels of the form

$$k(x) = p(x)j(x) + q(x), \tag{1.2}$$

where j satisfies a differential equation while, as in [6], p and q are polynomials. It is to consideration of this case that the present paper is devoted.

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One remark about (1.2). Although the restriction of p and q to polynomials seems rather severe, the situation is not as bad as it looks. The range of the argument $(x - t)$ of k in (1.1) is the *finite* interval $(a - b, b - a)$. Consequently, just so long as $k(x)$ is given by (1.2) with p and q continuous, the kernel can be successfully approximated by (1.2) with p and q polynomials. Thus, given, say, a singular kernel k, if the singularity alone is of Latta's type (*e.g.*, $\log \mid x \mid$ or $\mid x \mid^{\nu-1}$ or, for that matter, $H_0^{(1)}(\mid x \mid)$), the equation can be solved approximately by writing k in the form (1.2) where j contains the singularity and by approximating p and q by polynomials.

The paper begins with a brief discussion of the reduction of an equation with a kernel (1.2) to an equation with a Latta kernel. In the rest of the paper, three examples are worked out: the first two have logarithmic singularities; the last has the singularity $\mid x \mid^{\nu-1}$.

2. The idea of the method. Consider now (1.1) with $k(x)$ as in (1.2). It will be convenient to follow Latta and use operational notation—thus, we set

$$\int_a^b k(x - t)f(t)\,dt = \Lambda f$$

$$\int_a^b j(x - t)f(t)\,dt = \Gamma f,$$

where the kernel of Γf is the j of (1.2).

Now, the polynomial q of (1.2) is immaterial to the analysis; if $q(x) = \sum q_n x^n$, we have

$$\int_a^b q(x - t)f(t)\,dt = \sum q_n \int_a^b (x - t)^n f(t)\,dt$$

$$= \sum q_n^* x^n,$$

where the constants q_n^* depend on the moments $\int_a^b t^\nu f(t)\,dt$, $\nu = 1, \cdots, n$ of f. Thus, the term involving $q(x)$ can be incorporated into the function $\varphi(x)$ of (1.1); after the resulting equation has been solved for arbitrary values of the q_n^*, the solution can be substituted back into (1.1) to determine them. We may write, therefore, without loss of generality,

$$k(x) = p(x)j(x),$$

in place of (1.2).

Set $p(x) = \sum p_n x^n$. We then have

$$\Lambda f = \sum_n p_n \sum_\nu (-1)^\nu \binom{n}{\nu} x^{n-\nu} \int_a^b j(x - t)t^\nu f(t)\,dt$$

$$= \sum_n p_n \sum_\nu (-1)^\nu \binom{n}{\nu} x^{n-\nu} \Gamma x^\nu f. \tag{2.1}$$

Now, Latta's condition that $j(x)$ must satisfy a differential equation allows one to relate $\Gamma x^\nu f$ (and, as a consequence, $\Gamma x^\nu f$) to Γf by a linear differential equation. Using this fact, one can derive from (2.1) a differential equation relating Γf and Λf. However, Λf is known by (1.1) to be equal to $\varphi(x)$. Solving this differential equation between Γf and Λf will thus result in an equation

$$\psi(x) = \Gamma f \tag{2.2}$$

which must be satisfied if (1.1) is. The new equation (2.2) can then be solved by Latta's method since its kernel, j, satisfies his assumptions.

This will now be illustrated by examples.

3. First example. For our first example, we set

$$k(x) = x \log | x |. \tag{3.1}$$

This kernel itself seems to be of Latta's type, since it satisfies a differential equation ($xk'' - 1 = 0$) of the appropriate kind. However, $k(x)$ is non-singular; consequently, we may expect that no solution will exist unless the function φ of (1.1) is properly chosen. Since this fact violates a second assumption made in [4] and [5]—that there is a unique solution—Latta's method cannot be applied directly. We have chosen (3.1) as our first example in spite of the apparently complicating factor of possible non-existence of a solution because of the simplicity of the corresponding relation between Λf and Γf.

To begin, consider (3.1). With the notation of (1.2), we have $j(x) = \log | x |$, so that

$$(x - t)j'(x - t) - 1 = 0$$

($' = d/dx$). As discussed in [5], this implies

$$\Gamma' xf = x\Gamma' f - \mu_0 , \tag{3.2}$$

where μ_0 is the zero order moment of f. Now, obviously,

$$\Lambda f = x\Gamma f - \Gamma xf.$$

Therefore,

$$\begin{aligned} \Lambda' f &= x\Gamma' f + \Gamma f - \Gamma' xf \\ &= \Gamma f + \mu_0 , \end{aligned}$$

using (3.2). Thus, if f satisfies (1.1) with $k(x)$ as in (3.1), it must also satisfy (2.2), where

$$\psi = \varphi' - \mu_0 . \tag{3.3}$$

Equation (2.2) can be solved by Latta's method if $\varphi(x)$ is a polynomial or an exponential polynomial (φ is restricted to this class of functions in [4] and [5]) or, for general φ, by Carleman's formula ([7]; see also [6]). As an example, we choose $\varphi(x) = x$, $b = -a = 1$. By (2.2) and (3.3), we then have to solve

$$\Gamma f = 1 - \mu_0 . \tag{3.4}$$

Since $1 - \mu_0$ is a constant, the solution, according to Carleman's formula, say, is

$$f = \frac{\mu_0 - 1}{\pi \log 2} (1 - x^2)^{-1/2}. \tag{3.5}$$

Integrating, we can find μ_0 from the equation

$$\begin{aligned} \mu_0 &= \frac{\mu_0 - 1}{\pi \log 2} \int_{-1}^{1} (1 - x^2)^{-1/2} \, dx \\ &= \frac{\mu_0 - 1}{\log 2}. \end{aligned}$$

Indeed,

$$\mu_0 = \frac{1}{1 - \log 2},$$

and

$$f = \frac{(1 - x^2)^{-1/2}}{\pi(1 - \log 2)}.$$

That this really is a solution can be verified by substitution into the integral equation.

4. Second example. Now, let

$$k(x) = (1 + \alpha x) \log | x |. \tag{4.1}$$

Equation (1.1) with $k(x)$ as in (4.1) and $\varphi(x) = 1$ was solved for small α by Pearson in [6].

Here again we have $j(x) = \log | x |$, so that (3.2) continues to hold. Also,

$$\Lambda f = (1 + \alpha x)\Gamma f - \alpha \Gamma x f,$$

so that

$$\begin{aligned} \Lambda' f &= (1 + \alpha x)\Gamma' f + \alpha \Gamma f - \alpha \Gamma' x f \\ &= \Gamma' f + \alpha \Gamma f + \alpha \mu_0 , \end{aligned}$$

using (3.2). Thus, in this case, f can only satisfy (1.1) if it also satisfies (2.2), where

$$\psi' + \alpha \psi = \varphi' - \alpha \mu_0 . \tag{4.2}$$

It remains, then, to solve (2.2), where ψ is some solution of (4.2).

We now consider Pearson's example, setting $\varphi(x) = 1$, $b = -a = 1$, and solving (1.1) for small α. Equation (4.2) gives

$$\begin{aligned} \psi(x) &= -\mu_0 + Ae^{-\alpha x} \\ &= (A - \mu_0) - A\alpha x + O(\alpha^2), \end{aligned}$$

where A is a constant.

Define $f_0(x)$ by the equation

$$\Gamma f_0 = 1, \tag{4.3}$$

so that

$$f_0(x) = -\frac{1}{\pi \log 2}(1 - x^2)^{-1/2}$$

(see (3.4, 5)). Of course,

$$\begin{aligned} \Gamma' x f_0 &= -\int_{-1}^{1} f_0(x)\, dx \\ &= \frac{1}{\log 2} , \end{aligned}$$

by (3.2), so that

$$\Gamma x f_0 = \frac{x}{\log 2},$$

since xf_0 is an odd function.

We wish to consider

$$\begin{aligned} \Gamma f &= \psi \\ &= (A - \mu_0) - A\alpha x + O(\alpha^2) \\ &= (A - \mu_0)\Gamma f_0 - (A\alpha \log 2)\Gamma x f_0 + O(\alpha^2). \end{aligned} \tag{4.4}$$

Since the equation $\Gamma f = \psi$ has a unique solution [7] whose singularities can only be of the form $(1 - x^2)^{-1/2}$ at worst [5], it follows that

$$\begin{aligned} f &= (A - \mu_0)f_0 - (A\alpha \log 2)xf_0 + O[\alpha^2(1 - x^2)^{-1/2}] \\ &= (1 - x^2)^{-1/2}\left[\frac{\mu_0 - A}{\pi \log 2} + \frac{A\alpha}{\pi} x + O(\alpha^2)\right]. \end{aligned} \tag{4.5}$$

The constant A can now be evaluated by integrating (4.5). We obtain

$$\mu_0 \equiv \int_{-1}^{1} f\, dx = \frac{\mu_0 - A}{\log 2} + O(\alpha^2),$$

so that

$$A = \mu_0(1 - \log 2) + O(\alpha^2)$$

and

$$f = (1 - x^2)^{-1/2}\left[\frac{\mu_0}{\pi} + \frac{\mu_0\alpha}{\pi}(1 - \log 2)x + O(\alpha^2)\right]. \tag{4.6}$$

To evaluate μ_0, recall that we must have

$$\begin{aligned} 1 &= \Lambda f \\ &= (1 + \alpha x)\Gamma f - \alpha \Gamma x f. \end{aligned} \tag{4.7}$$

Now, (4.6) is equivalent to

$$f = -(\mu_0 \log 2)f_0 - (\mu_0\alpha \log 2)(1 - \log 2)xf_0 + O[\alpha^2(1 - x^2)^{-1/2}].$$

Hence,

$$\begin{aligned} \Gamma f &= -(\mu_0 \log 2)\Gamma f_0 - (\mu_0\alpha \log 2)(1 - \log 2)\Gamma x f_0 + O(\alpha^2) \\ &= -\mu_0 \log 2 - \mu_0\alpha(1 - \log 2)x + O(\alpha^2) \end{aligned}$$

by (4.3) and (4.4). Also,

$$\begin{aligned} \Gamma x f &= -(\mu_0 \log 2)\Gamma x f_0 + O(\alpha) \\ &= -\mu_0 x + O(\alpha). \end{aligned}$$

Therefore, (4.7) results in

$$\begin{aligned} 1 &= (1 + \alpha x)[-\mu_0 \log 2 - \mu_0\alpha(1 - \log 2)x] + \mu_0\alpha x + O(\alpha^2) \\ &= -\mu_0 \log 2 + O(\alpha^2), \end{aligned}$$

so that

$$\mu_0 = -\frac{1}{\log 2} + O(\alpha^2),$$

giving finally,

$$f = (1 - x^2)^{-1/2}\left[-\frac{1}{\pi \log 2} - \frac{\alpha}{\pi}\left(\frac{1}{\log 2} - 1\right)x + O(\alpha^2)\right],$$

which agrees with Pearson's result.

5. Final example. For our final example, we choose

$$k(x) = (1 + \alpha x^2) \mid x \mid^{\nu-1} + 1, \qquad 0 < \nu < 1;$$

this will illustrate a kernel with a non-logarithmic j and a non-zero q [see (1.2)]. The integral equation is

$$\varphi(x) = \int_a^b [1 + \alpha(x - t)^2] \mid x - t \mid^{\nu-1} f(t)\, dt + \int_a^b f(t)\, dt. \tag{5.1}$$

The last term in this equation is a constant, μ_0, and so if we set

$$\varphi_* = \varphi - \mu_0, \tag{5.2}$$

we have

$$\varphi_*(x) = \int_a^b k_*(x - t) f(t)\, dt$$

$$\equiv \Lambda_* f,$$

where

$$k_*(x) = (1 + \alpha x^2) \mid x \mid^{\nu-1}.$$

Now, if $j(x) = \mid x \mid^{\nu-1}$,

$$xj' + (1 - \nu)j = 0,$$

and so if $\Gamma f = \int_a^b j(x - t) f(t)\, dt$,

$$\Gamma' xf = x\Gamma' f + (1 - \nu)\Gamma f. \tag{5.3}$$

Also,

$$\Lambda_* f = (1 + \alpha x^2)\Gamma f - 2\alpha x \Gamma x f + \alpha \Gamma x^2 f. \tag{5.4}$$

We obtain from (5.3) that

$$\Gamma'' x^2 f = x^2 \Gamma'' f + 2(2 - \nu) x \Gamma' f + (1 - \nu)(2 - \nu)\Gamma f; \tag{5.5}$$

therefore, differentiating (5.4) and using (5.3) and (5.5), we obtain

$$\Lambda_*'' f = \Gamma'' f + \alpha\nu(1 + \nu)\Gamma f.$$

Thus, in order to solve (5.1), we must first consider the equation

$$\Gamma f = \psi, \tag{5.6}$$

where ψ is a solution of

$$\psi'' + \alpha\nu(1 + \nu)\psi = \varphi_*'' .$$

Equation (5.6) can be solved by Latta's method if ψ (and, therefore, φ) is an exponential polynomial. The details of the computations, though complicated, are not impossible, but it would serve no purpose to complete them here.

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ON THE FIRST STABILITY INTERVAL OF THE HILL EQUATION*

By C. R. PUTNAM (*Purdue University*)

Let λ denote a real parameter and let $f = f(t)$ be a real-valued, continuous periodic function of period 1. It is known (Liapounoff) that the Hill equation

$$x'' + (\lambda + f(t))x = 0 \qquad [' = d/dt, f(t + 1) = f(t)] \tag{1}$$

is stable for $\lambda = 0$, so that every solution of the equation $x'' + f(t)x = 0$ is bounded, whenever

$$f \geq 0, \qquad f \not\equiv 0 \quad \text{and} \quad \int_0^1 f\,dt \leq 4; \tag{2}$$

see, e.g., [1], [6]. Moreover the constant 4 of (2) is the best possible in the sense that (2) is not sufficient for the stability of $x'' + fx = 0$ if the 4 is replaced by $4 + \epsilon$ (ϵ, a positive constant) [3]. If λ_0 and λ_1 denote respectively the left and right end-points of the first stability interval of (1) then the first two conditions of (2) imply $\lambda_0 < 0$ while all conditions together imply $\lambda_1 > 0$ (and so $\lambda = 0$ is interior to the first interval of stability of (1)). Actually the inequality $\lambda_1 > 0$ is implied by the single condition

$$\int_0^1 f^+\,dt \leq 4, \quad \text{where} \quad f^+(t) = \max\,[0, f(t)] \tag{3}$$

(see [6]); moreover, the estimate

$$\lambda_1 > 4 - \int_0^1 f^+\,dt = 4\left(1 - \frac{1}{4}\int_0^1 f^+\,dt\right) . \tag{4}$$

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easily follows. The object of this note is to obtain a "best possible" refinement of (4), namely, *if* (3) *holds, then*

$$\lambda_1 \geq \pi^2\left(1 - \frac{1}{4}\int_0^1 f^+\,dt\right), \tag{5}$$

where the equality holds only if $f \equiv 0$.

It is known that when $\lambda = \lambda_1$, Eq. (1) possesses a "half-periodic" solution $x = y(t)$ satisfying $y(t + 1) = -y(t)$; see, e.g., [5]. In case $f \equiv 0$, this function is $\sin \pi t$ which satisfies $x'' + \pi^2 x = 0$ (i.e., (1) for $f \equiv 0$ and $\lambda = \lambda_1 = \pi^2$); thus the equality of (5) holds in this case. It is clear then that the italicized assertion above can become false if the π^2 of (5) is replaced by $\pi^2 + \epsilon$, even if $f \not\equiv 0$. That the assertion can become false if the $\frac{1}{4}$ of (5) is replaced by $\frac{1}{4} - \epsilon$ is obvious from an earlier remark concerning the stability criterion furnished by (2).

In order to prove the italicized assertion, let $y(t)$ denote the half-periodic solution of (1) for $\lambda = \lambda_1$ considered above, so that $y'' + [\lambda_1 + f(t)]y = 0$, and let a, $a + 1$ denote two zeros of y. A multiplication by y of both sides of the last equation, followed by an integration and an application of the inequality $f \leq f^+$, leads to

$$\lambda_1 \int_a^{a+1} y^2\,dt \geq \int_a^{a+1} y'^2\,dt - \int_a^{a+1} f^+ y^2\,dt. \tag{6}$$

Since $2y(t) = \int_a^t y'\,ds - \int_t^{a+1} y'\,ds$, it follows that $2\,|\,y(t)\,| \leq \int_a^{a+1} |\,y'\,|\,ds$ and hence, by the Schwarz inequality, that $4y^2(t) \leq \int_a^{a+1} y'^2\,dt$. Consequently

$$\int_a^{a+1} f^+ y^2\,dt \leq \frac{1}{4}\int_a^{a+1} f^+\,dt \int_a^{a+1} y'^2\,dt;$$

hence, by (6),

$$\lambda_1 \int_a^{a+1} y^2\,dt \geq \left(1 - \frac{1}{4}\int_0^1 f^+\,dt\right)\int_a^{a+1} y'^2\,dt. \tag{7}$$

In view of (3), which implies $\lambda_1 > 0$, and the Wirtinger inequality $\pi^2 \int_a^{a+1} y^2\,dt \leq \int_a^{a+1} y'^2\,dt$, where the equality holds only if $y = \text{const.} \sin \pi(t - a)$ (see, e.g., [2], p. 184), relation (7) implies (5) and the proof of the italicized assertion is now complete.

Remark. The referee has pointed out an interesting parallelism between the inequality (5) and an inequality in a recent paper by J. Peetre [4], see pp. 16–17, for the least eigenvalue of an eigenvalue problem for partial differential equations with a Riemann metric.

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A STABILITY CRITERION FOR QUASI-HARMONIC VIBRATIONS*

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1. Let $\omega = \omega(t)$ be a continuous function which is periodic, choose its period to be π, and call

$$x'' + \omega^2(t)x = 0 \tag{1}$$

a *stable* oscillator if all solutions $x(t)$ (i.e., if two, linearly independent solutions) stay bounded as $| t | \to \infty$. It is understood that $\omega(t)$ is assumed to be real-valued, which means that $\omega^2(t) \geqq 0$. Here the sign of equality is allowed (for certain t-values, which can be isolated or form intervals), but it will be assumed that $\omega(t)$ does not vanish identically, i.e., that the case $x \equiv 0$ of (1) is excluded.

According to Liapounoff [1], (1) *must* be stable if

$$\int_0^\pi \omega^2(t)\, dt < 4/\pi. \tag{2}$$

But it is also known [2] that if $4/\pi$ is replaced by any numerical constant greater than $4/\pi$, then the assumption which thus results from (2) will *allow* the instability of (1) (for *certain* frequency functions $\omega(t)$, to be chosen in an appropriate manner). In other words, $4/\pi$ is the best absolute constant in (2).

This is not, of course, at variance with the fact that if $\omega(t)$ is the constant 1, then (1) is stable but (2) is violated (since $\pi > 4/\pi$). But it can be expected that the absolute constant $4/\pi$ will be improved if it is assumed that $\omega(t)$ keeps away from 0 by a fixed amount. In this regard, a result of P. S. Gusarova [3] states that if

$$\omega(t) \geqq 1, \tag{3}$$

then

$$\int_0^\pi \omega^2(t)\, dt \leqq 12/\pi, \tag{4}$$

a relaxation of (2), will guarantee the stability of (1).

2. This appears to clear up the situation in a satisfactory manner. Actually, the stability criterion (3)–(4) is unsatisfactory from the point of view of most applications. For, from such a point of view, the stability of (1) ought to be ensured by imposing a limitation on a characteristic which measures the relative degree of the non-constancy of $\omega(t)$, rather than on a characteristic measuring the amount by which $\omega(t)$ keeps away from 0. But the criterion (3)–(4) is of this latter kind.

Needless to say, the criterion of the desired type will also have to involve a limitation of the mean-value of $\omega^2(t)$ over a period; see (2) or (4). In the desideratum, the latter limitation, supplied by the bounds $4/\pi$, $12/\pi$ in the respective criteria (2), (3)–(4) of Sec. 1, will have to be expressed in terms of the characteristic measuring the non-constancy of $\omega(t)$.

3. The simplest choice of such a characteristic is the quotient

$$\vartheta = (M - m)/(M + m), \tag{5}$$

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where

$$m = \min_{0 \leq t \leq \pi} \omega^2(t), \qquad M = \max_{0 \leq t \leq \pi} \omega^2(t), \tag{6}$$

hence

$$0 \leqq m \leqq M < \infty \quad \text{and} \quad M > 0 \tag{7}$$

(in fact, if $M = 0$, then (1) reduces to the unstable, but trivial, equation $x'' = 0$, excluded since the beginning of Sec. 1). It is clear from (6) and (7) that the quotient (5) exists and is subject to the inequalities

$$0 \leqq \vartheta < 1; \tag{8}$$

and that (1) is the closer to the state of a harmonic oscillator ($\vartheta = 0$, $\omega(t) = \text{const.} > 0$) the closer is ϑ to 0. For this reason, the following criterion, the deduction of which is the purpose of this note, is along the lines of the desideratum formulated in Sec. 2:

If the coefficient function of an oscillator (1) *does not vanish identically and is of period* π, *then all solutions* $x(t)$ *of* (1) *stay bounded, as* $\mid t \mid \to \infty$, *whenever the mean-value of the squared "instantaneous frequency"* $\omega(t)$ *satisfies the inequality*

$$\frac{1}{\pi} \int_0^\pi \omega^2(t)\, dt < Q(1 - \vartheta^2)^{1/2}, \tag{9}$$

where ϑ *denotes the characteristic of* $\omega(t)$ *defined by* (5) *and* (6), *and* Q *is a certain numerical constant,*

$$Q = 0.8043 \cdots \tag{10}$$

It will remain undecided whether (10) is the best absolute constant allowed in this stability criterion. But it is seen from (10) that the value of the product πQ certainly exceeds the value of $4/\pi$. Hence, if ϑ is reasonably small, then the criterion (9) is more favorable than Liapounoff's criterion (2).

Note that, the quotient (5) being a *relative* ("percentile") characteristic of $\omega(t)$, nothing like an *absolute* limitation (3) is involved in (9).

4. The sufficiency of (9) for the stability of (1) will be concluded from the following criterion, proved by Borg [4] as a counterpart of Liapounoff's criterion (quoted at the beginning of Sec. 1): If $f(t)$ is a real-valued, continuous function of period T, and if $f(t)$ is non-negative throughout but does not vanish identically, then all solutions $x(t)$ of

$$x'' + f(t)x = 0 \tag{11}$$

will stay bounded (as $\mid t \mid \to \infty$) whenever the coefficient of (11) is restricted by the condition

$$\int_0^T f(t)\, dt < 64P^4/3T^3, \tag{12}$$

where

$$P = \int_0^{1/2\pi} (1 + \sin^2 \varphi)^{-1/2}\, d\varphi. \tag{13}$$

In order to identify (11) with (1), let $f(t) = \omega^2(t)$ and $T = \pi$. Then (12) can be written in the form

$$\frac{1}{\pi}\int_0^\pi \omega^4(t)\,dt < 64P^4/3\pi^4. \tag{14}$$

Consequently, it is sufficient to show that (9) is implied by (14).

5. First, if $f_1(t)$ and $f_2(t)$, where $u \leqq t \leqq v$, are arbitrary continuous functions satisfying the inequalities

$$0 < m_i \leqq f_i(t) \leqq M_i\,, \quad \text{where} \quad i = 1, 2,$$

then, by "the (partial) converse of Schwarz's inequality" [5], the quotient

$$4\int_u^v f_1^2(t)\,dt\cdot\int_u^v f_2^2(t)\,dt\Big/\left[\int_u^v f_1(t)f_2(t)\,dt\right]^2$$

cannot exceed the square of

$$(M_1M_2/m_1m_2)^{1/2} + (m_1m_2/M_1M_2)^{1/2}.$$

If this is applied to

$$f_1(t) = \omega^2(t), \qquad f_2(t) = 1 \quad \text{and} \quad u = 0, v = \pi,$$

it is seen from (6) that

$$4\pi\int_0^\pi \omega^4(t)\,dt \leqq [(m/M)^{1/2} + (M/m)^{1/2}]^2\left[\int_0^\pi \omega^2(t)\,dt\right]^2.$$

Consequently, (14) is certainly satisfied if

$$\frac{1}{\pi}\int_0^\pi \omega^2(t)\,dt < \frac{16P^2/\pi^2}{3^{1/2}[(m/M)^{1/2} + (M/m)^{1/2}]}. \tag{15}$$

In view of the end of Sec. 4, this means that the proof will be complete if it is shown that (9) is implied by (15).

6. It is readily verified from (5) that

$$(m/M)^{1/2} + (M/m)^{1/2} = 2/(1 - \vartheta^2)^{1/2}.$$

Hence (15) can be written in the form (9) if Q is defined by

$$Q = 2^3\cdot 3^{-1/2}P^2/\pi^2. \tag{16}$$

Finally, the substitution $\alpha = \sin\varphi$ transforms (13) into

$$P = \int_0^1 (1 - \alpha^4)^{-1/2}\,d\alpha. \tag{17}$$

But (17) is the celebrated constant of Stirling [6], the complete lemmiscatic integral (of first kind). Since

$$P = 1.3110\cdots, \tag{18}$$

(10) follows from (16).

7. As an illustration, consider Mathieu's equation

$$x'' + (a + b\cos 2t)x = 0, \tag{19}$$

where a, b are constants. This is an equation of the form (1), with a positive $\omega^2(t)$ of period π and of mean-value a, if $0 < b < a$. Since (6) reduces to $m = a - b$ and $M = a + b$ in this case, it is seen from (5) that

$$1 - \vartheta^2 = 4c/(1 + c)^2, \quad \text{where } c = b/a.$$

Hence, condition (9) becomes $a < Qc^{1/2}/(1 + c)$.

Since $c = b/a$, the latter inequality can be written in the form

$$a/b < 4Q^2/(a + b)^2, \quad \text{where} \quad 0 < b < a. \tag{20}$$

Consequently, Mathieu's equation (19) is stable (and, what is more, "disconjugate") whenever its parameters a, b satisfy the conditions (20), where Q is the value (10), defined by (16) and (17).

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A HEAT CONDUCTION PROBLEM INVOLVING A SPECIFIED MOVING BOUNDARY*

By R. E. GIBSON (*Imperial College, London*)

1. Introduction. Physical problems governed by an equation of heat conduction type occasionally arise which require a condition to be satisfied on a boundary moving at a specified rate. In the problems with which we are concerned the movement of the boundary is generated by the accretion there of new material and not by deformation of the medium. Typical examples are: the dissipation of excess pore water pressure in a clay layer upon which fresh material is being deposited [1], and the type of thermal problem considered by Benfield [2] relating to the theory of the formation of the earth on the dust cloud hypothesis. The moving boundary is also a feature of thermal problems where a change in state of the conducting medium occurs [3]. Here, the motion of the boundary separating the phases is not given and must be determined as part of the solution. We can hope, therefore, that our problems, where the motion of the boundary is given, will prove to be less intransigent.

While working on one-dimensional problems of this type, we found that exact solutions could be obtained in two cases, namely, when the rate of movement of the boundary is proportional to $t^{-1/2}$ and when it is constant. Although in few actual problems may these conditions be realized, we believe our results to be of interest since it appears unlikely that closed solutions can be obtained for an arbitrary motion of the boundary.

2. Problem with spherical symmetry. We consider first a case with spherical

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symmetry. For $t < 0$ in $0 \leq r < \infty$ the temperature is constant, $v(r, t) = v_0$. At $t = 0$ the formation of a sphere begins, within which an homogeneous rate of heat production $A(t)$ per unit volume occurs. Additional material, at a temperature v_0, arriving on the surface of the sphere also participates in the generation of heat at a rate $A(t)$, the current radius of the sphere being $R(t)$.

The appropriate equation for the temperature is

$$\frac{1}{\kappa}\frac{\partial v}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{A(t)}{K}, \qquad 0 \leq r \leq R(t), \tag{1}$$

where the constants κ and K are respectively the diffusivity and thermal conductivity of the medium. The problem is to determine $v(r, t)$ for $t > 0$ subject to the conditions

$$v[R(t), t] = v_0, \tag{2}$$

$$\frac{\partial v}{\partial r}[0, t] = 0. \tag{3}$$

3. Separation of variables. If in (1) we take as new variables $u = r(v - v_0)$, $\xi = r/R(t)$ we obtain

$$\kappa\frac{\partial^2 u}{\partial \xi^2} + \frac{\kappa A}{K}\xi R^3 = R^2\frac{\partial u}{\partial t} - \xi RR'\frac{\partial u}{\partial \xi}, \tag{4}$$

where the prime denotes differentiation with regard to t. The boundary of the sphere is now "fixed" at $\xi = 1$. We now investigate the conditions for which

$$u = f(t)G(\xi) + \xi h(t) \tag{5}$$

is a solution of (4). This requires, firstly, that

$$\frac{\kappa}{K}A(t) = R^{-2}(h'R - hR'). \tag{6}$$

The variables f and G are found to be separable only if $R = ct^{1/2}$, and then (7)

$$\frac{1}{G}\left[\frac{\kappa}{c^2}\frac{d^2G}{d\xi^2} + \frac{\xi}{2}\frac{dG}{d\xi}\right] = tf'/f = \lambda, \tag{8}$$

where λ is an arbitrary constant. From the second of (8)

$$f = Dt^{\lambda}, \tag{9}$$

where D is an arbitrary constant. Now, in terms of the new variables the boundary conditions (2) and (3) are

$$f(t)G(1) + h(t) = 0, \tag{10}$$

$$f(t)G(0) = 0. \tag{11}$$

From (9) and (11) we have

$$h(t) = -DG(1)t^{\lambda}$$

and therefore, from (5)

$$u = Dt^{\lambda}[G(\xi) - \xi G(1)]. \tag{12}$$

The constant D is found from (6) to be given by

$$\frac{\kappa c}{K} A(t) = -DG(1)t^{\lambda-3/2}(\lambda - \tfrac{1}{2}). \tag{13}$$

The rate of heat production is therefore restricted to the form

$$A(t) = A_0 t^s, \qquad s > -1. \tag{14}$$

From (12)–(14) we then have

$$u = \frac{\kappa A_0}{K} \frac{ct^{3/2+s}}{(s+1)} \left[\xi - \frac{G(\xi)}{G(1)}\right].$$

The function G is now found from (8), one of the arbitrary constants arising being determined from (11). Finally, we arrive at the following expression for the temperature,

$$v = v_0 + \frac{\kappa A_0}{K} \frac{t^{s+1}}{(s+1)} \left[1 - \frac{{}_1F_1\left(s + \frac{5}{2}, \frac{3}{2}, \frac{r^2}{4\kappa t}\right)}{{}_1F_1\left(s + \frac{5}{2}, \frac{3}{2}, \frac{c^2}{4\kappa}\right)} \exp - \frac{1}{4}\left(\frac{r^2}{\kappa t} - \frac{c^2}{\kappa}\right)\right], \tag{15}$$

which expression may be reduced to a finite combination of elementary functions if either s or $(s + \frac{1}{2})$ is an integer [4]. In particular, for a constant rate of heat generation $(s = 0)$

$$v = v_0 + \frac{\kappa A_0}{K}\left(\frac{c^2 t - r^2}{c^2 + 6\kappa}\right). \tag{16}$$

4. Integral equation. In the following section we derive a solution to (1), valid for an arbitrary rate of heat generation, when the motion of the boundary is

$$R(t) = \mu t.$$

We take

$$\kappa A(t) = KF'(t)$$

and consider a solution of (1) of the type

$$v = v_0 + F(t) - r^{-1}t^{-1/2} \int_0^\infty \rho^{-1} g(\rho) \left[\exp - (r - \rho)^2/4\kappa t - \exp - (r + \rho)^2/4\kappa t\right] d\rho. \tag{17}$$

Apart from a constant factor the third term represents the temperature at (r, t) due to an initial temperature $g(r)$ in the region $r \geq 0$ when heat generation is zero everywhere. Since the sphere vanishes at $t = 0$ no initial condition on the temperature need be satisfied, and we are therefore at liberty to regard g as an arbitrary function which must be chosen so that

$$v[R(t), t] = v_0, \qquad t > 0$$

and this requirement is met with if g satisfies

$$\tfrac{1}{2}R(t)F(t)t^{1/2} \exp \frac{R^2}{4\kappa t} = \int_0^\infty \frac{g(\rho)}{\rho} \sinh \frac{\rho R}{2\kappa t} \exp - \frac{\rho^2}{4\kappa t} d\rho. \tag{18}$$

In the special case* when $R = \mu t$ that part of the kernel involving the hyperbolic function may be absorbed in the unknown function. The following changes of variable

$$\rho^2 = \zeta, \qquad t = 1/4\kappa p$$

then reduce (18) to the form

$$\int_0^\infty H(\zeta)e^{-p\zeta}\,d\zeta = \frac{\mu}{16\kappa^{3/2}p^{3/2}}F\left(\frac{1}{4\kappa p}\right)\exp\frac{\mu^2}{16\kappa^2 p}, \tag{19}$$

where

$$H(\zeta) = \frac{g(\zeta^{1/2})}{2\zeta}\sinh\frac{\mu\zeta^{1/2}}{2\kappa}.$$

Proceeding formally from (19) we find that

$$g(\rho) = \frac{\mu\rho^2}{8\kappa^{3/2}}\operatorname{cosech}\frac{\mu\rho}{2\kappa}\cdot\frac{1}{2\pi i}\int_{B\tau} p^{-3/2}F\left(\frac{1}{4\kappa p}\right)\exp\left(\frac{\mu^2}{16\kappa^2 p} + \rho^2 p\right)dp$$

which, together with (17), leads to an expression for the temperature satisfying the governing equation (1) and the boundary conditions (2) and (3). In particular, when heat generation takes place at a constant rate

$$F = A_0 t/K$$

the temperature is given by

$$v = v_0 + \frac{A_0}{K}\left[t - \frac{2}{\mu^2 r}\left(\frac{\kappa}{\pi t}\right)^{1/2}\exp - \frac{r^2}{4\kappa t}\right.$$
$$\left.\cdot\int_0^\infty \rho\sinh\frac{\rho r}{2\kappa t}\left\{\frac{\mu\rho}{2\kappa}\coth\frac{\mu\rho}{2\kappa} - 1\right\}\exp - \frac{\rho^2}{4\kappa t}\,d\rho\right]. \tag{20}$$

5. One dimensional problems. Problems one-dimensional in some other coordinate system may be treated along lines similar to those discussed in the preceding sections.

For example, the temperature within an infinite cylinder of radius $R = ct^{1/2}$, when heat is generated according to (14), is found to be

$$v = v_0 + \frac{\kappa A_0}{K(s+1)}t^{s+1}\left[1 - \frac{{}_1F_1(s+2, 1, r^2/4\kappa t)}{{}_1F_1(s+2, 1, c^2/4\kappa)}\exp - \frac{1}{4}\left(\frac{r^2}{\kappa t} - \frac{c^2}{\kappa}\right)\right]$$

while for the conditions considered in Sec. 4 the temperature is given by

$$v = v_0 + F(t)$$
$$- \frac{1}{4\pi i\kappa t}\exp - \frac{r^2}{4\kappa t}\int_0^\infty\int_{B\tau}\frac{\rho}{p}F\left(\frac{1}{4\kappa p}\right)\frac{I_0\left(\frac{r\rho}{2\kappa t}\right)}{I_0\left(\frac{\mu\rho}{2\kappa}\right)}\exp\left(\frac{\mu^2}{16\kappa^2 p} + p\rho^2 - \frac{\rho^2}{4\kappa t}\right)d\rho\,dp,$$

wherein

$$F(t) = \frac{\kappa}{K}\int_0^t A(\tau)\,d\tau.$$

*The solution of Sec. 3 can also be attained directly from (18) after suitable changes of variable and the use of the convolution theorem.

6. Numerical evaluation. In the spherical problem the quantities of physical importance are the temperature at the centre and the gradient at the boundary. When heat is generated at a constant rate and the radius increases according to (7), these quantities are found directly from (16). We note, firstly, that the temperature difference $[v(0, t) - v_0]$ is a constant fraction $1/(1 + 6\kappa c^{-2})$ of the same difference if no leakage of heat across the boundary is allowed, and, secondly, as time goes on the temperature gradient at the boundary increases without limit.

On the other hand when $R = \mu t$ this gradient approaches an asymptotic value $-\kappa A_0/\mu K$ which may be seen by considering the behaviour of

$$\frac{\mu K}{\kappa A_0}\frac{\partial v}{\partial r}[R, t] = \frac{2}{T} - 1 - \frac{8}{\pi^{1/2}T^{1/2}}e^{-T/4}\int_0^\infty y^3 e^{-y^2}\operatorname{cosech} yT^{1/2}\cdot dy$$

as

$$T(=\mu^2 t/\kappa) \to \infty .$$

The infinite integral has been evaluated numerically along the lines suggested by Goodwin [5], and the relation between the gradient and the time factor T is shown in Fig. 1.

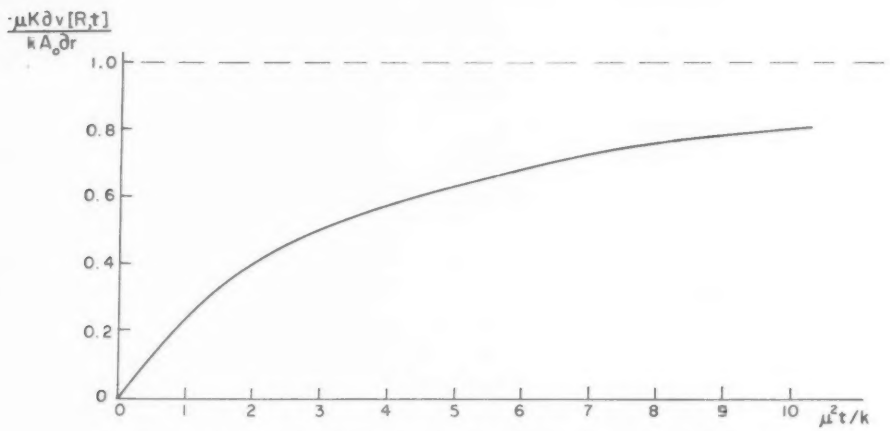

FIG. 1. Variation of boundary temperature gradient with time. The dashed line represents the asymptotic value.

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ON A GENERALIZATION OF A RESULT OF WINTNER*

By RICHARD BELLMAN (*The Rand Corporation*)

In a recent note [1] Wintner proved the following interesting result.

Theorem 1. *Consider the two equations*

$$\text{(a)}\quad u'' + f(t)u = 0, \qquad \text{(b)}\quad v'' + g(t)v = 0. \tag{1}$$

If there exist two linearly independent solutions of (1a), u_1 *and* u_2, *such that*

$$\int^{\infty} (| u_1 |^2 + | u_2 |^2) | f - g | \, dt < \infty \tag{2}$$

then every solution of (1b) *can be written in the form*

$$v = c_1 u_1 + c_2 u_2 + o(| u_1 | + | u_2 |). \tag{3}$$

This is an extension of known stability results, see [2], to which it reduces if we assume that all solutions of (1a) are bounded as $t \to \infty$.

Let us now show that we can obtain a generalization of this result following the method used in our book [2], to establish the Hukuwara stability theorem, of which this will be an extension.

Theorem 2. *Consider the vector-matrix systems*

$$\text{(a)}\quad \frac{dx}{dt} = A(t)x, \qquad \text{(b)}\quad \frac{dy}{dt} = B(t)y. \tag{4}$$

Let $X(t)$ *be the solution of*

$$\frac{dX}{dt} = A(t)X, \qquad X(0) = I. \tag{5}$$

If

$$\int^{\infty} || B(t) - A(t) || \, || X(t) || \, || X^{-1}(t) || \, dt < \infty \tag{6}$$

then every solution of (4b) *may be written*

$$y = Xc + o(|| X ||) \tag{7}$$

as $t \to \infty$.

The norms of matrices and vectors are taken to be respectively $\sum_{i,j} | x_{ij} |$ and $\sum_i | x_i |$.

Proof. Write

$$\frac{dy}{dt} = A(t)y + [B(t) - A(t)]y. \tag{8}$$

Then, if $y(0) = b$, we have

$$y = X(t)b + \int_0^t X(t)X^{-1}(s)[B(s) - A(s)]y \, ds. \tag{9}$$

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Hence

$$|| y || \leq || X(t) || \, || b || + \int_0^t || X(t) || \, || X^{-1}(s) || \, || B(s) - A(s) || \, || y || \, ds. \tag{10}$$

Thus, if we set

$$\begin{aligned} u(t) &= || X^{-1}(t) || \, || B(t) - A(t) || \, || y(t) ||, \\ v(t) &= || B(t) - A(t) || \, || X(t) || \, || X^{-1}(t) ||, \end{aligned} \tag{11}$$

we obtain the scalar inequality

$$u \leq c_1 v + v \int_0^t u \, ds. \tag{12}$$

This yields, as a consequence of the fundamental inequality, [2], or directly, the estimate

$$\int_0^t u \, ds \leq c_1 \int_0^t v(s) \exp \left[\int_s^t v \, dr \right] ds. \tag{13}$$

By assumption $\int^\infty v \, ds < \infty$. Hence the integral

$$\int_0^\infty X^{-1}(s)[B(s) - A(s)]y \, ds \tag{14}$$

converges. This means that we can write (9) in the form

$$y = X(t)b + X(t) \int_0^\infty X^{-1}(s)[B(s) - A(s)]y \, ds - X(t) \int_t^\infty X^{-1}(s)[B(s) - A(s)]y \, ds \tag{15}$$

which yields the stated result.

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BOUNDS ON THE ERROR IN THE UNIT STEP RESPONSE OF A NETWORK*

By PAUL CHIRLIAN (*New York University*)

Summary. Bounds have been placed upon the error, in the unit step response of a network, caused by a departure of the transfer function from its desired value. The theorems developed place bounds on the error when it is caused by:

1) A departure of the amplitude function from zero, for frequencies above a stated cut-off frequency.
2) Any deviation of the amplitude or phase function from its desired value.

These bounds are more readily evaluated than the actual errors, and thus prove useful in the design of networks.

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Introduction. In many analysis or synthesis procedures, the transfer function of a network only approximates the desired value. It is often necessary to know how this approximation affects the transient response. Existing methods require tedious analytic or graphic analysis to determine the deviation of the transient response. More readily evaluated bounds, which eliminate the need for this analysis would be useful. Such bounds have been developed in the theorems presented in the body of this paper.

The first considers the error, in the unit step response, caused by an assumption that the transfer function is zero above a given cut-off frequency.

The remaining theorems deal with the errors, in the unit step response, which result from deviation of either the amplitude or phase functions from their desired values.

Throughout this paper, the transfer function of the network in question will be written as $T(\omega)e^{i\theta(\omega)}$ where the amplitude function $T(\omega)$ is never negative and $T(0) > 0$. In addition the magnitude of the per cent error in the unit step response, referred to its final value will be written as $| \%\delta |$. That is, if $A(t)$ is the actual unit step response and $A_1(t)$ is the approximate unit step response, then

$$| \% \ \delta | = \frac{| A(t) - A_1(t) | \max \times 100}{T(0)}.$$

Bounds on the error in the unit step response caused by a band-limiting approximation. It is often convenient to assume that the transfer function of a network is zero for all values of frequency above a given cut-off frequency ω_c . This assumption, called a band limiting approximation, introduces an error into the calculated value of the unit step response. It is desirable to bound this error so that the approximation can be justified. This bound can be obtained by evaluating an integral of the form

$$\int_{\omega_c}^{\infty} \frac{T(\omega)}{T(0)} \frac{1}{\omega} \, d\omega.$$

This evaluation may be simplified by replacing $T(\omega)/T(0)$ by a new, more readily integrated function $G(\omega)$, where $G(\omega) \geq T(\omega)/T(0)$ for all $\omega \geq \omega_c$ and $\int_{\omega_c}^{\infty} [G(\omega)/\omega] d\omega$ exists. The best bound on the error will be obtained when $G(\omega) = T(\omega)/T(0)$ and this procedure should always be considered as a first choice.

Theorem 1 will bound the error caused by the band limiting approximation.

Theorem 1. *If* $T(\omega)/T(0) \leq G(\omega)$ *for* $\omega > \omega_c$, *then*

$$| \% \ \delta | \leq \frac{100\%}{\pi} \int_{\omega_c}^{\infty} \frac{G(\omega)}{\omega} \, d\omega.$$

Proof. For any network the unit step response is given by

$$A(t) = \tfrac{1}{2}T(0) + \frac{1}{\pi} \int_0^{\infty} \frac{T(\omega)}{\omega} \sin [\omega t + \theta(\omega)] \, d\omega.$$

Thus in this case the error is given by

$$| \% \ \delta | = \frac{100\%}{\pi} \left| \int_{\omega_c}^{\infty} \frac{T(\omega)}{\omega T(0)} \sin [\omega t + \theta(\omega)] \, d\omega \right|.$$

The bound is then obtained by utilizing the fact that $T(\omega)/T(0) \leq G(\omega)$ and $| \sin [\omega t + \theta(\omega)] | \leq 1$. Theorem 1 will now be applied to a particular function which falls off as $1/\omega^n$, where $n > 0$. This is of use in many commonly encountered networks. If $G(\omega) = \epsilon(\omega_c/\omega)^n$ the following is obtained.

Corollary 1a. *If* $T(\omega)/T(0) \leq \epsilon(\omega_c/\omega)^n$ *for* $\omega > \omega_c$ *and* $n > 0$ *then*

$$| \% \delta | \leq \frac{100\epsilon\%}{n\pi}.$$

Some typical values given by this bound are

If $\epsilon = 0.1$ and $n = 1$ then $| \%\delta | \leq 3.18\%$

If $\epsilon = 0.707$ (3 db down) and $n = 3$, then $| \%\delta | \leq 7.5\%$.

Bounds on the error in the unit step response caused by an approximation of the amplitude or phase function. When an arbitrary transfer function is not realized exactly, but is only approximated, an error appears in the unit step response.

The following theorems place bounds upon this error. The first of these deals with an approximation of the amplitude function. In this case the transfer function of the desired network is $T(\omega)e^{j\theta(\omega)}$. Often this transfer function is approximated by the transfer function $[T(\omega) + T_\epsilon(\omega)] \exp [j\theta(\omega)]$. The amplitude error function $T_\epsilon(\omega)$ is a real function of ω which may be positive and negative. Then if $\int_0^\infty | T_\epsilon(\omega) |/\omega \, d\omega$ exists, $| \%\delta |$ can be bounded.

Theorem 2[1]. *If the amplitude error function* $T_\epsilon(\omega)$ *exists, then*

$$| \% \delta | \leq \frac{100\%}{\pi} \int_0^\infty \left| \frac{T_\epsilon(\omega)}{T(0)} \right| \frac{1}{\omega} \, d\omega.$$

Proof. This theorem is proven by formally writing the expression for the error and then utilizing the fact that $| \sin [\omega t + \theta(\omega)] | \leq 1$.

If a function $M(\omega)$ can be found such that $| T_\epsilon(\omega)/T(0) | \leq M(\omega)$ for all ω and the term $\int_0^\infty [M(\omega)/\omega] \, d\omega$ is more readily evaluated than the integral of Theorem 2, then it may prove convenient to replace $| T_\epsilon(\omega)/T(0) |$ by $M(\omega)$ in this theorem; the bound developed in this case will be weaker than the one originally presented. However, in some cases, it may be expedient to use the simpler integral to obtain a rough estimate of the error.

The next two theorems will place a bound upon the error in the unit step response produced by an approximation of the phase function. The transfer function of the desired network is $T(\omega) \exp [j\theta(\omega)]$. Many times this transfer function is approximated by $T(\omega) \exp \{j[\theta(\omega) + \phi(\omega)]\}$. The phase error function $\phi(\omega)$ is a real function of ω which may be positive and negative. If the integrals

$$\int_0^\infty \frac{T(\omega)}{T(0)} \left| \frac{\phi(\omega)}{\omega} \right| d\omega \quad \text{and} \quad \int_0^\infty \frac{T(\omega)}{T(0)} \frac{[\phi(\omega)]^2}{\omega} \, d\omega$$

exist then $| \%\delta |$ may be bounded.

Theorem 3. *If the phase error function* $\phi(\omega)$ *exists, then*

$$| \% \delta | \leq \frac{100\%}{\pi} \int_0^\infty \frac{T(\omega)}{T(0)} \frac{1}{\omega} \left\{ 2 \sin^2 \left[\frac{\phi(\omega)}{2} \right] + | \sin \phi(\omega) | \right\} d\omega$$

$$| \% \delta | \leq \frac{100\%}{\pi} \int_0^\infty \frac{T(\omega)}{T(0)} \frac{1}{\omega} \left\{ \frac{[\phi(\omega)]^2}{2} + | \phi(\omega) | \right\} d\omega.$$

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[1]The proof of this theorem is similar to one given by A. H. Zemanian, *An approximate method of evaluating integral transforms*, J. Appl. Phys. **25**, 262-266 (Feb. 1954)

Proof. The first bound is obtained by formally evaluating the error and then utilizing the fact that $| \sin [\omega t + \theta(\omega)] | \leq 1$ and $| \cos [\omega t + \theta(\omega)] | \leq 1$. The second equation is obtained from the first by realizing that $| \sin x | \leq | x |$. The first of these equations will provide a smaller bound than the second. However, the second equation is more readily evaluated and if $\phi(\omega)$ is small, in the range of integration, then the results obtained from this second equation can be quite satisfactory.

If the normalized amplitude function $T(\omega)/T(0)$ is itself bounded, the results of Theorem 3 are readily extended to obtain a bound which is more easily evaluated.

Corollary 3a. If $T(\omega)/T(0) \leq M$ *then*

$$| \% \ \delta | \leq \frac{100\ M\%}{\pi} \int_0^\infty \frac{1}{\omega} \left\{ 2 \sin^2 \left[\frac{\phi(\omega)}{2} \right] + | \sin \phi(\omega) | \right\} d\omega,$$

and

$$| \% \ \delta | \leq \frac{100\ M\%}{\pi} \int_0^\infty \frac{1}{\omega} \left\{ \frac{[\phi(\omega)]^2}{2} + | \phi(\omega) | \right\} d\omega.$$

When networks whose amplitude functions are such that Theorems 1 or 1a indicate that a negligible error is introduced by neglecting the response at frequencies greater than ω_c then the upper limit of integration can be changed from ∞ to ω_c in the equation of Theorems 3 and 3a. This technique should especially be used when the effects of phase correction is to be determined since extremely large phase errors usually occur at frequencies above ω_c . These phase errors usually have little bearing on the transient response.

Conclusion. The theorems which have been developed here place bounds on the error, in the unit step response, caused by a deviation of the transfer function from its desired value.

In transient analysis procedures, it is often convenient to neglect the response of a network above a given cut-off frequency. Theorems 1 and 1a bound the error produced by such procedure. In addition, knowledge of results obtained in Theorems 1 and 1a can simplify the design of a network, since these results indicate a frequency above which the response is no longer important. Theorems 2, 3, and 3a bound the error caused by a deviation of either the amplitude or phase function from its desired value. Deviation of this type will occur in synthesis procedures, where the desired transfer function cannot be exactly realized with a finite number of realizable elements.

When a network is designed to produce a desired unit step response, the error of the transfer function, in itself, has no direct significance. However, this is often the known error. Theorems 2, 3, or 3a can then be employed to relate the error in either the amplitude or phase function, to the error in the unit step response.

The use of these theorems, therefore, can provide valuable information to the network designer.

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ON AN INITIAL VALUE PROBLEM CONCERNING TAYLOR INSTABILITY OF INCOMPRESSIBLE FLUIDS*

By G. F. CARRIER AND C. T. CHANG (*Harvard University*)

Abstract. By taking the effect of surface tension and viscosity into consideration, Bellman and Pennington [2] have generalized the original treatment of the problem of Taylor instability [1]. They claim, however, that a problem in which the motion starts from rest cannot be treated with their linearized formulation. In this paper, the initial value problem is treated and, although the algebra becomes more complicated, the linearized analysis suffices. In particular, the cut-off wave number as found in [2] is not modified.

1. Introduction. The problem of Taylor instability [1], asks about the instability of a small disturbance imposed on the interface separating two infinitely extended fluids when the system is undergoing a constant acceleration directed from the lighter towards the heavier fluid.

The original treatment was limited to incompressible nonviscous fluids. Surface tension effects at the interface were ignored. Referring to a simple sinusoidal initial disturbance of amplitude a and wave number k, the result showed that the time history of the surface disturbance with respect to the accelerated frame of reference is given by

$$\eta(x, t) = a \cosh (nt) \cos kx. \tag{1.1}$$

The growth factor "n" is related monotonically to the wave number "k" by the expression

$$n^2 = -\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g^* k. \tag{1.2}$$

In the above, ρ_1 , ρ_2 are the densities of the lighter and the heavier fluid respectively, while

$$g^* = g + g_1 \tag{1.3}$$

with g as the usual gravitational constant, and g_1 , the imposed acceleration.

Bellman and Pennington [2], later generalized Taylor's original result by taking the effect of surface tension and viscosity into consideration. They assumed a standing wave type solution and obtained the following expression for the growth factor "n"

$$n^2 = -\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g^* k - \frac{T}{\rho_2 + \rho_1} k^3, \tag{1.4}$$

where T is the surface tension of the interface. Equation (1.4) indicates the existence of a cut-off wave number k_c , i.e. a wave number such that no instability arises when $k > k_c$, where

$$k_c = \left[-(\rho_2 - \rho_1) \frac{g^*}{T} \right]^{1/2}. \tag{1.5}$$

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Their result also shows that viscosity tends to retard the growth of the interface. However, they note a dilemma, for according to their solution it is impossible to start a motion from rest. They attributed this seeming paradox to the linearization performed in their analysis.

The object of this paper is to show that this paradox is easily resolved. As a consequence, the time history of the interface is given by a more complicated expression than that of Eq. (1.1); in particular, it is no longer possible to define a simple growth factor. However, this algebraic complication, neither compromises the existence of the cut-off wave number nor changes its value as given by Eq. (1.5).

2. Formulation of the problem. For simplicity, we shall confine our attention to an air-liquid system, i.e. we may approximate the density ratio by $\rho_1/\rho_2 = 0$. The more general problem can be treated in precisely the same way. The flow field is taken as the half plane bounded above by the free surface $y = \eta(x, t)$. The medium is assumed to be incompressible but viscous; surface tension effects are to be included. The whole body of the fluid is assumed to undergo a constant vertical acceleration g_1 directed from air towards the liquid. Referring to a stream function ψ, the linearized momentum equation with respect to the accelerated frame of reference can be written as

$$\frac{\partial}{\partial y}\frac{\partial}{\partial t}[\nabla^2 \psi - \nu \nabla^2 \nabla^2 \psi] = 0 \tag{2.1}$$

and ψ itself must satisfy

$$\frac{\partial}{\partial t}\nabla^2 \psi = \nu \nabla^2 \nabla^2 \psi. \tag{2.2}$$

The coordinate axes are chosen in such a way that the x-axis is taken in the unperturbed plane of the free surface while the y-axis is taken normal to it.

With some amount of algebraic manipulation it can be shown that ψ must satisfy the following boundary conditions (they imply $\sigma_{yy} = \sigma_{xy} = 0$) at the free surface:

$$\begin{aligned} \nu(\psi_{xxx} - \psi_{xyy})_t - \psi_{xtt} + g^*\psi_{xy} &= -\frac{T}{\rho}\psi_{xxy}, \\ -\psi_{xxy} + \psi_{yyy} &= 0, \quad \text{at} \quad y = 0. \end{aligned} \tag{2.3}$$

The usual kinematic requirement at the free surface is

$$\eta_t = -\psi_{xy}. \tag{2.4}$$

With no loss in generality, we can restrict the analysis to a single Fourier component in x; thus we may write

$$\begin{aligned} \psi(x, y, t) &= \phi(y, t)e^{ikx}, \\ \eta(x, t) &= \sigma(t)e^{ikx}. \end{aligned} \tag{2.5}$$

We may also define

$$\begin{aligned} f(y, s) &= \int_0^\infty e^{-st}\phi(y, t)\,dt \\ \mu(s) &= \int_0^\infty e^{-st}\sigma(t)\,dt \end{aligned} \tag{2.6}$$

and obtain in place of Eqs. (2.2) and (2.3) the following equations

$$\left[\frac{d^2}{dy^2} - k^2\left(1 + \frac{s}{\nu k^2}\right)\right]\left[\frac{d^2}{dy^2} - k^2\right]f = -\frac{1}{\nu}\left[\frac{d^2}{dy^2} - k^2\right]\phi(y, 0), \qquad (2.7)$$

$$\text{for} \quad -\infty < y < 0,$$

and

$$k^2 f_v + f_{vvv} = 0$$

$$s[k^2 f + f_{vv}] + s^2 f + \left(g^* + \frac{T}{\rho}k^2\right)f_v \qquad (2.8)$$

$$= s\phi(0, 0) - \frac{i}{k}\left(g^* + \frac{T}{\rho}k^2\right)\sigma(0), \quad \text{at} \quad y = 0.$$

From these expressions one observes that the problem is determined theoretically once the initial surface disturbance and the initial velocity field throughout the domain are prescribed.

3. Illustrative example. To illustrate our discussion, let us take

$$\psi(x, y, 0) = 0 \qquad (3.1)$$

and

$$\eta(x, 0) = a \cos kx, \qquad (3.2)$$

i.e. the motion is started from rest with an initial disturbance imposed at the free surface in the form of a sinusoidal wave having amplitude a and wave number k.

Using the foregoing method, it can be verified that η_t is given by

$$\eta_t = -a\left\{\nu k^2 \beta \sum_{i=1}^{4} A_i \alpha_i \exp\left[(\alpha_i^2 - 1)k^2 t\right] \operatorname{erfc}\left[-\alpha_i(\nu k^2 t)^{1/2}\right]\right\} \cos kx \qquad (3.3)$$

where α_i are the roots of the quartic

$$P(\tau) = \tau^4 + 2\tau^2 - 4\tau + (1 + \beta) \qquad (3.4)$$

with

$$\tau^2 = 1 + \frac{s}{\nu k^2}$$

the quantity A_i is given by

$$A_i = \left(\frac{dP}{d\tau}\right)^{-1}_{\tau = \alpha_i} \qquad (3.5)$$

and the parameter β is given by

$$\beta = \frac{T}{\nu^2 \rho k_c}\left[\left(\frac{k_c}{k}\right) - \left(\frac{k_c}{k}\right)^3\right]. \qquad (3.6)$$

Here, k_c is the cut-off wave number corresponding to $\rho_1/\rho_2 = 0$, i.e.

$$k_c = \left\{-\rho\frac{g^*}{T}\right\}^{1/2}. \qquad (3.7)$$

Equation (3.4) indicates that for each j the Re $(\alpha_j^2) < 1$ when $| k/k_c | \geq 1$, but this inequality does not hold when $| k/k_c | < 1$. Thus we are in agreement with the principal result obtained in [2] but note that the model of that paper has broader validity than was claimed.

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Comments on the paper

WAVES PRODUCED BY A PULSATING SOURCE TRAVELLING BENEATH A FREE SURFACE*

Quarterly of Applied Mathematics **XV**, 249–255 (1957)

By PAUL KAPLAN (*Stevens Institute of Technology*)

There appears to be an error in the analysis performed in the subject paper by Dr. Tan [1], which has led to erroneous conclusions in regard to the character of the resulting wave pattern and the asymptotic conditions at infinity that are used to arrive at a unique solution. Other treatments of this same problem have been carried out by Becker [2], Kaplan [3], and Wu [4], and they have arrived at different conclusions from those of Tan. Their results show two harmonic wave trains on the downstream side of the source when $\tau > \frac{1}{4}$, corresponding to the roots K_3 and K_4, in agreement with the results of Tan. However, when $0 < \tau < \frac{1}{4}$, Tan indicated that four harmonic wave trains were on the downstream side, with no disturbance found to propagate to infinity upstream. This led to his conclusion that the imposition of an asymptotic upstream condition of "vanishing disturbance at infinity" is sufficient to render his solution unique. The results of Becker, Kaplan and Wu were all in agreement, and showed that when $0 < \tau < \frac{1}{4}$ there were three harmonic wave trains on the downstream side, and one harmonic wave train on the upstream side. It can be shown that the wave in the upstream side corresponds to the root K_2 and that the error in Tan's analysis was not recognizing the fact that, in the notation of [1],

$$\lim_{\beta \to 0} ImK_2 = 0- \qquad (0 < \tau < \tfrac{1}{4})$$

not $0+$, as he indicated.

If this error were corrected, and the resulting analysis carried out in the same manner as done in the paper, the results would have been in agreement with that of the other authors. The imposition of the requirement of no disturbance upstream would not be valid, but the use of the technique of Rayleigh's artifice in order to secure uniqueness would not be affected, in spite of the artificial nature of this factor. A more realistic

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approach, which has more physical appeal, is to formulate the problem as an unsteady initial value problem with the motion assumed to start from a state of rest, and the steady state oscillatory solution is obtained when the time $t \to \infty$ and "transients" have disappeared. The behavior of the waves at infinity, i.e. the radiation conditions, are determined directly from that type of analysis by only imposing boundedness conditions at infinity, without any *a priori* assumptions about the character of the waves. This technique has been utilized by Kaplan and Wu in their treatments of this problem.

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BOOK REVIEWS

(*Continued from p. 396*)

Statistical analysis of stationary time series. By Ulf Grenander and Murray Rosenblatt. John Wiley & Sons, Inc., New York, and Almqvist & Wiksell, Stockholm, 1957. 300 pp. $11.00.

The basic probability model considered in this monograph consists of a discrete parameter stochastic process $y_t = x_t + m_t$, where x_t is a stationary stochastic process and m_t is a linear combination of known regression vectors. Often in the book x_t is specialized to a linear process, which can be described roughly as an infinite linear combination of indepent identically distributed random variables. Since x_t is stationary, the convariance sequence $r_n = E(x_t , x_{t+n})$ depends only on an n. The hermitian matrix (r_{m-n}) is non-negative definite, so by a well-known theorem in Fourier series there exists a nondecreasing function $F(\lambda)$ which has $r_0 , r_1 , \cdots$, as its Fourier-Stieltjes coefficients. This function is called the spectral distribution function (sdf) of x_t . The central problem of the book is to study the distributions of certain statistical estimators of the sdf and of its derivative, the spectral density (sd). These estimators are quadratic functions of a finite partial realization of the process y_t . The estimation of the regression coefficients and tests of hypotheses concerning them are also considered. The continuous parameter case is also discussed in several places. Some of the material in the later chapters is here published for the first time.

The first three chapters of the book contain background material. The first chapter presents the basic spectral theory of stationary processes. The second chapter contains a discussion of the least square theory of prediction, interpolation, and estimation of regression constants, when the spectrum is known. The language of linear manifolds and function spaces is used to introduce the fundamental ideas. In the next chapter some examples of the older approaches to time series are given. The models considered are finite parameter ones such as the model in the first paragraph above but with independent, identically distributed stochastic elements x_t , and also the autoregressive case in which a finite moving average of the elements y_t is set equal to a succession of independent, identically distributed random variables. A feature of this chapter consists of a number of carefully polished proofs of some of the leading classical results including Slutzky's theorem.

The authors start taking up the new approaches in the fourth chapter, in which statistical estimators of a continuous sd are introduced and their asymptotic biases and variances are derived. Emphasis is placed on quadratic estimators which are weighted averages of the periodogram ("spectrograph estimates"). For linear processes they are shown to be as good in a certain sense for estimating the sd at a particular point as any of the more complicated types of quadratic estimators. There is a rigorous discussion of the asymptotic bias and variance of a certain estimator of the sd in the time-continuous case. This is one of the few major demonstrations presented for that case in the book.

In the next chapter some sketches of the role of spectral analysis in the study of electronic devices, turbulence, and water waves are presented. In the sixth chapter, distribution theory is again taken up. The discussion there centers upon a linear process and upon an estimator of the sdf which consists merely of the indefinite integral of the periodogram. The major result can be described roughly as follows: Under certain regularity conditions, the asymptotic distribution function, say, $H(a)$, of the maximum distance (properly scaled) between the estimator $F^*(\lambda)$ and the sdf $F(\lambda)$ for $0 \leqq \lambda \leqq \pi$ is equal to the probability that a certain normal process $z(\lambda)$ stays inside $(-a,a)$ for $0 \leqq \lambda \leqq \pi$. The covariance function of $z(\lambda)$ unfortunately depends upon the sdf, so it is difficult to use the result to set up asymptotic confidence bands except in special cases. Extensions to more general estimators and to the case where a regression is present are considered. The chapter concludes with numerical illustrations in which the estimation techniques are applied to some artificially generated numerical series.

Various questions arising in regression analysis are considered in the seventh chapter. There is a comparison of the asymptotic efficiency of the least square estimator and the minimum variance estimator. This is followed by a careful study of least square estimation. The exposition is somewhat intricate and depends upon a device for handling the regression vectors all at once which the authors call the regression spectrum. The last chapter deals with miscellaneous problems. Perhaps the most interesting consists of the problem of prediction when the spectrum must be estimated. The authors present a consciously non-rigorous derivation of the mean square error of prediction in this case. The book con-

cludes with some problems for the reader and an appendix which presents some of the classical results connected with the names of Hardy and Littlewood on functions analytic in the unit circle.

The stated prerequisite for reading this book is "a knowledge of statistics and basic probability theory equivalent to that contained in H. Cramer, "Mathematical Methods of Statistics." " With this understanding, the exposition is highly self-contained. However, the student who has reached the level of mathematical competence at which he can just comfortably read Cramer's book, but nothing more difficult, will find some rough going here. To feel really at home, he will need to have had a good deal of experience with the methods of rigorous mathematical analysis, and in particular he should have a reasonably close acquaintance with the modern theory of trigonometric series. The proofs which the authors give for previously known results are often elegant but tend to be unmotivated and highly condensed.

The study of time series has always presented many challenging and urgent problems. The authors claim, with considerable justification, that the highly specified models used in the past have often been unsatisfactory. They embark upon a program designed to investigate whether the spectral methods which have been so useful in the engineering sciences and in physics can be developed in such a way as to give a better theory for general purposes. The investigation is undoubtedly of great importance, but as with every other known approach to time series analysis, there are some discouraging facts to be faced. For example, knowledge of the spectrum is an end in itself in many engineering and physical problems, and this is not always true in other contexts. The statistical estimation of the spectrum is somewhat tricky technically, particularly if not too many data are at hand. (For instance, the relation between the sample size and the maximum lag of the products used in the estimator must be carefully adjusted.) Accurate point estimates of the sd are elusive. The spectrum of a stationary process does not uniquely determine the process without further assumptions, so even if one could estimate the spectrum of a given series exactly, not all questions about its structure would be cleared up. The random elements of the processes encountered in the social sciences may be generally non-stationary, and the adaptation of spectral methods to non-stationary series is still somewhat problematical. The analytical derivation of exact sampling distributions in the spectral approach may be very difficult. (The statistical results in the present monograph are almost all of an asymptotic nature.)

These problems are all openly recognized by the authors, and they are careful to point out the tentative character of parts of their monograph. They state in the preface that one of their main purposes is "to stimulate research in time series analysis which will lead to practically useful and theoretically sound methods." The reviewer feels that the publication of this book is an important milepost in the development of the mathematical theory of time series. The book is rigorous and accurate, and at the same time it is full of interesting results and techniques which will probably insure its continuing value as a reference whichever way the research in the field turns.

There are the usual minor misprints which characterize a first edition. Perhaps the only accident which might confuse the reader takes place on the bottom line of page 86 and page 87, where the x's should all be y's.

J. H. Curtiss

Vector spaces and matrices. By Robert M. Thrall and Leonard Tornheim. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1957. xiii + 318 pp. $6.75.

This book presents the theory of vector spaces and matrices simultaneously at two levels, one concrete and one axiomatic, the concrete approach sometimes preceding, sometimes following, the axiomatic one. The authors have succeeded in providing a text that aims to develop in the student conceptual grasp rather than mere manipulative skill, as well as to aid him in the applications of the subject to various branches of pure and applied mathematics. The exposition, throughout clear and concise, is more detailed in the chapters forming roughly the first half of the book (often more so than is usual), and more advanced in the later ones. Each chapter contains a good number of selected exercises which frequently supplement preceding theory. Among the features generally not found in texts on this subject is a development of the general theory of simple algebraic extensions of a field (chapter 8) and an introduction to linear inequalities and game theory (chapter 11).

Table of contents: 1. Vector spaces. 2. Linear transformations and matrices. 3. Systems of linear

equations. 4. Determinants. 5. Equivalence relations and canonical forms. 6. Functions of vectors. 7. Orthogonal and unitary equivalence. 8. Structure of polynomial rings. 9. Equivalence of matrices over a ring. 10. Similarity of matrices. 11. Linear inequalities. There are appendices on mathematical induction, relations and mappings, and a bibliography.

H. A. ANTOSIEWICZ

Water waves—the mathematical theory with applications. By J. J. Stoker. Interscience Publishers, New York and London, 1957. xxi + 567 pp. $12.00.

The theory of water waves as a branch of mathematical hydrodynamics may be said to begin with Laplace's work on tides, and during the nineteenth century it attracted the attention of many of the most eminent physicists. Their work is described by Lamb who devotes nearly one-third of his book to it. After 1900 the subject became unfashionable, interest in wave motion shifted to electromagnetism and atomic physics, while interest in the application of hydrodynamics shifted to aerodynamics; only a few mathematicians kept working on the numerous problems left unsolved in this field. Some outstanding theoretical work was done during this period which is only now receiving its merited recognition. New activity began during the recent war, and is still continuing. Much new and interesting mathematics was discovered, although much still remains to be done. The progress in theoretical knowledge has been accompanied by advances in hydraulics. A leading part in these developments has been played by the Institute of Mathematical Sciences of New York University where the well-known author of the present work is Professor of Mathematics.

This book is the first connected account of the subject since Lamb's *Hydrodynamics*, but so rapid has been the expansion of the subject that a book of nearly 600 pages cannot now cover all topics, and the author has been faced with the difficult problem of selection. The contents of the book are as follows:

Part I
1. Basic hydrodynamics.
2. The two basic approximate theories (small-amplitude and long-wave)

Part II Subdivision A. Waves simple harmonic in time.
3. Simple harmonic oscillations in water of constant depth.
4. Waves maintained by simple harmonic surface pressure in water of uniform depth. Forced oscillations.
5. Waves on sloping beaches and past obstacles.

Subdivision B. Motions starting from rest. Transients.
6. Unsteady motions.

Subdivision C. Waves on a running stream. Ship waves.
7. Two-dimensional waves on a running stream in water of uniform depth.
8. Waves caused by a moving pressure point. Kelvin's theory of the wave pattern created by a moving ship.
9. The motion of a ship, as a floating rigid body, in a seaway.

Part III
10. Long waves in shallow water.
11. Mathematical hydraulics.

Part IV 12. Problems in which force surface conditions are satisfied exactly. The breaking of a dam. Levi-Civita's theory.

As is explained in the introduction, the problems included in the book are chosen mostly from among those on which the author or his colleagues at New York University have worked, and this gives the book a personal character. Completeness is not aimed at, even in the problems treated here, and the literature cited is confessedly incomplete, even as regards recent work. The mathematical and physical equipment needed by readers differs in different parts of the book, and some of the most interesting sections (e.g. on ship wave patterns) may be read with little previous knowledge of hydrodynamics. There is great variety in the problems. "The physical problems range from discussion of wave motion over sloping beaches to flood waves in rivers, the motion of ships in a sea-way, free oscillations of enclosed bodies of water such as lakes and harbors, and the propagation of frontal discontinuities in the atmosphere, to mention just a few. The mathematical tools employed comprise just about the

whole of the tools developed in the classical linear mathematical physics concerned with partial differential equations, as well as a good part of what has been learned about the non-linear problems of mathematical physics. Thus potential theory and the theory of the linear wave equation, together with such tools as conformal mapping and complex variable methods in general, the Laplace and Fourier transform techniques, methods employing a Green's function, integral equations, etc. are used. The non-linear problems are of both elliptic and hyperbolic type." But, in spite of the diversity of the material, a thread of continuity runs through the book. Friction is neglected except in the chapter on mathematical hydraulics.

In the space allowed for this review it is not possible to discuss in detail any section of the book. and short comments on a few topics must suffice. One-half of the book treats small-amplitude waves, for which the non-linear free-surface condition is replaced by a linearized approximation. In the division dealing with time-periodic waves, the section on sloping beaches does not claim to describe adequately the behavior of real waves on beaches, the interest lies in the mathematical boundary-value problem, and in the behavior of solutions near the corner. The literature on this problem alone is now so big that only a few of the new methods could find a place in the book (There is a section on breaking waves in Chapter 10). Among other problems treated in this part of the book is Sommerfeld's problem of diffraction by a half-plane for which a new and very pretty solution is given. Many other methods and applications of time-periodic linearized waves receive brief mention. The next subdivision, devoted to transients, contains among other things a discussion of boundary conditions at infinity (radiation conditions) and of the principle of stationary phase. The next subdivision, on waves in a running stream, gives a treatment of unsteady waves, leading to the appropriate radiation condition, and goes on to describe ship wave patterns due to pressure points moving along straight lines and circles, with photographic illustrations. The last chapter in this subdivision treats ships in a seaway in a systematic manner. The infinitely thin ship (Michell ship) and its limitations are more clearly explained here than anywhere else in the literature, and there is reason to believe that this will be the opening of a new phase of development in this neglected interesting and technically important subject. A similar treatment for planing ships and other ships of thin section is also given. (For a recent extension of this work see A.S. Peters and J. J. Stoker, Comm. Pure Appl. Math, 10, 1957, 399-490.) The third part of the book deals with long-wave theory; in Chapter 10 an interesting variety of problems is treated; in Chapter 11 friction is regarded as dominant, and the theory is applied to flood routing on the Mississippi with success, using parameters determined from earlier floods and progressively corrected. Such calculations can be made quickly enough to give useful predictions about the later stages of a flood. Comparison with other methods of calculation would have been interesting.

The last part of the book discusses exact solutions, and gives a new proof of the existence of periodic progressive waves on deep water, first established by Levi-Civita.

The reviewer believes that this book has completely achieved its aim. A picture emerges of the present state of the subject: many problems have been solved, but some of the most interesting major problems remain unsolved, including the following:

(1) In the linearized theory of time-periodic waves (Laplace's equation with the simplest mixed-type boundary condition) the uniqueness theory is still missing.
(2) The theory of unsteady non-linear wave motion is still very incomplete, with corresponding difficulty in the interpretation of experiments.
(3) The development of experimental techniques, and the detailed comparison of theory and experiment still leave much to be desired

F. URSELL

Turbolenza di Parete. By Carlo Ferrari. Levrotto & Bella, Torino, 1957. 116 pp.

It is now generally recognized that the inner and outer parts of the turbulent boundary layer have quite different turbulent structure, and as Professor Ferrari shows in his lecture notes, acceptance of this single premise is almost sufficient for rejuvenating the mixing-length theory. Indeed, the inner sublayer is dominated by the wall shear to such an extent that dimensional argument and Taylor series yield quite detailed information, and the assumption of complete self-preservation in the outer layer (critically examined by Townsend, "Structure of Turbulent Shear-flow," Cambr. Univ. Press, 1956)

then suffices for deducing the logarithmic distributions of mean velocity and skin friction. On the further assumption of a turbulent shear diffusivity constant across the outer layer (cf. again Townsend), the mean velocity distribution can be completed, and this is carried out in detail for channel flow and the boundary layer without, and finally with, pressure gradient.

Of course, little understanding of turbulent structure is gained, but on the other hand, it is possible to make the theory appear almost independent of assumptions derived from experiments. In fact, no experiments are mentioned, beyond the Laufer-Townsend energy balance of the boundary layer without pressure gradient. An advantage of this approach is that the calculations can be extended in a natural way also to heat transfer and compressibility, in the manner pioneered by Reichardt and Ferrari, and this extension is presented for the case without pressure gradient.

R. E. MEYER

Quantum mechanics. By F. Mandl. Academic Press, Inc., New York, and Butterworths Scientific Publications, London, 1957. x + 267 pp. $6.50.

This is a second edition of a justly popular book. A final chapter on the Dirac equation, requiring a rudimentary knowledge of the special theory of relativity, has been added to the first edition, published in 1954.

Apart from this, there are no essential changes. The exercises, together with hints for their solution, remain a most valuable part of the text.

RICHARD BELLMAN

The numerical solution of two-point boundary problems in ordinary differential equations. By L. Fox. Oxford at the Clarendon Press, 1957. xi + 371. $9.60.

This book presents the most complete treatment of the two-point boundary problem by numerical methods which has so far appeared. The nearest rival is Collatz's "Numerische Behandlung" which devotes about 100 pages to boundary problems for ordinary differential equations. The points of view of Collatz and Fox differ considerably, the former leaning more toward analytical, the latter more toward purely numerical methods.

The scope of Fox's book is shown by an examination of the contents. Chapter 1 gives a brief introduction to the general problem, and emphasizes the distinction between two-point and initial-point problems, a distinction which is far more serious for numerical methods than for purely analytical methods. Chapter 2 presents finite differences and difference equations, supplying the basic computational background needed in later chapters. In Chapter 3, the author studies the solution of algebraic equations, both linear and non-linear, by direct methods, by iteration, and by relaxation. The method of relaxation is then applied to simultaneous finite-difference equations.

Up to this point the material has been preparatory. In Chapter 4 the author takes up second order differential equations with two-point boundary conditions. Various types of boundary conditions are treated by a variety of methods. Chapter 5 is entitled "First-order equations," and the reader may wonder what goes on, since the solution of a first-order equation cannot satisfy two independent conditions. A somewhat over-simplified explanation is the following. Obviously $(y_{n+1} - y_{n-1})/2h$ is in general a better approximation to the first derivative y_n' than is $(y_{n+1} - y_n)/h$, where h is the step interval. But when the first expression above replaces the derivative in the differential equation we are led to a *second order* differential equation. This differential equation has an extraneous oscillatory solution which must be suppressed if possible. Nevertheless it may be easier to solve the second order difference equation by relaxation than to solve the original differential equation by step-by-step methods. Techniques for suppressing the oscillation are discussed.

In Chapter 6, equations of higher order are examined. If central differences are used (as is often desirable because of greater accuracy for a given number of terms) the same problem of extraneous oscillatory solutions bobs up whenever the order of the equation is odd. This phenomenon is related to questions of stability for certain methods of solving differential equations step-by-step.

Chapter 7 is devoted to eigenvalue problems associated with ordinary differential equations. When

a linear differential equation is replaced by a difference equation the eigenvalue problem is approximated by the problem of latent roots and vectors of a matrix. Hence Chapter 7 contains a rather full treatment of latent roots and vectors. In Chapter 8, Dr. Fox discusses various ways to use initial-value methods in order to solve two-point problems and eigenvalue problems, pointing out some advantages and some dangers of this procedure. In Chapter 9 he considers accuracy and precision of boundary value methods. In linear problems or "linearized" problems this involves essentially an examination of the latent roots of an appropriate matrix. Also involved are error estimates for the various formulas of finite differences used in the solution. The book concludes with miscellaneous methods and comments.

The text abounds in interesting worked examples which not only illustrate the procedures being explained but also reveal the author's skill and long practical experience in computational methods. As stated in the preface, the author has concentrated on methods "which are suitable for paper-and-pencil and desk-machine computation. Most of the methods are adaptable to automatic high-speed machines."

It is possible that the real strength of the book, namely its wealth of computational ideas, may be a disappointment to some readers who want to find a simple cut-and-dried procedure which can be immediately programmed for a high-speed digital computer. In fact the reviewer would like to have seen at least some space given to solutions on high speed machinery, even though he agrees with the author that a good grasp of paper-and-pencil methods should precede electronic methods.

All in all Dr. Fox has made a welcome and significant contribution to the literature of numerical methods.

W. E. Milne

Progress in semiconductors. By Alan F. Gibson, R. E. Burgess, and P. Aigrain. John Wiley & Sons, Inc., New York, 1956. Volume I, viii + 220 pp. $8.00. Volume II, vii + 280 pp. $10.50.

This consists at present of a two volume summary (Vol. I, 1956 and Vol. II, 1957) or review of work in semiconductor research and applications. It is intended that additional volumes be published annually. Each volume contains review articles written by specialists in the field. One of the outstanding features of these two volumes is the very large reference list accompanying each article. In view of the thousands of papers on semiconductor work published thus far, such a venture as this seems worthwhile even if only for the reference lists. The articles are, however, excellent whether for specialist or non-specialist. These annual volumes are particularly recommended for anyone who wants to be aware of the main features of semiconductor work without reading hundreds of papers.

R. Truell

An introduction to probability theory and its applications. Volume I. By William Feller, John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1957. xv + 461 pp. $10.75.

The first edition of this book, which is well known as a standard reference and popular textbook in probability theory, has been modified somewhat for this second edition. Although the style and the general outline of topics remain nearly the same, the detailed wording has been changed in many places. There is more emphasis on suggested topics for first reading and changes in wording appear to be motivated by a desire to improve the book's usefulness as a text. Despite the fact that many sections have been made more concise, the second edition is somewhat larger than the first as a result of the addition of two new chapters on "Fluctuations in coin tossing and random walk" and "Compound distributions, Branching processes." This edition also puts greater emphasis on waiting times and the connection between recurrent events and Markov chains. The high quality of the first edition is certainly maintained if not, in fact, considerably improved. The book's many admirers, however, continue to wait patiently for Volume II.

G. Newell

Mathematics and wave mechanics. By R. H. Atkin. John Wiley & Sons, Inc., New York, 1957. xv + 348 pp. $6.00.

This book consists of eight introductory chapters (198 pages) dealing with mathematical preparation for the study of quantum mechanics. The remaining six chapters deal with quantum mechanics and some of the more important applications. The form of the book should make it particularly valuable to seniors in physics, applied mathematics or chemistry, who want to begin the study of quantum mechanics. It will also be valuable to graduate students who lack mathematical background for the study of quantum mechanics.

R. Truell

Principles and techniques of applied mathematics. By Bernard Friedman. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1956. ix + 315 pp. $8.00.

This book has the laudable intention "to show how the powerful methods developed by the abstract [mathematical] studies can be used to systematize the methods and techniques for solving problems in applied mathematics." More specifically, the abstract theory of linear spaces and linear operators on these spaces is used extensively in an attempt to systematize the use of Green's function and eigenfunction expansions in the solution of ordinary and partial differential equations.

The first two chapters contain an excellent presentation of the theory of abstract linear spaces and operators which is required in the remainder of the work, and in this sense the book is self-contained. After introducing the Dirac δ-function in Chapter Three, the author introduces Green's function for a second order linear ordinary differential equation and expresses solutions in terms of the Green's function. The eigenvalue problem for ordinary differential equations is considered in Chapter Four, and Chapter Five discusses cursorily Laplace's equation, the diffusion equation, and the wave equation.

The reviewer feels that the strong emphasis on the theory of linear operators in the development of the theory of the last three chapters is obtained largely at the expense of rigour. For example, the proof of the completeness of the eigenfunctions $u_n = \sin nx$ $(n = 1, 2, \cdots)$ promised (on page 197) to be given later could not be found in the book by the reviewer. The formal approach, particularly in the last chapter, seems to obscure the underlying physical nature of the solutions to the equations considered. The reviewer also feels that the scope of the book is too limited for it to warrant such a broad title.

R. T. Shield

Proceedings of the Fifth Midwestern Conference on fluid mechanics. Edited by A. M. Kuethe. University of Michigan Press, Ann Arbor, 1957. viii + 388 pp. $8.00.

The Volume collects 26 papers, among which a discussion of low-density Couette flow by H. T. Yang and L. Lees stands out as clearly deserving careful reading by any students of the slip-flow regime. T. Y. Li calculates the hypersonic boundary layer corresponding to a prescribed pressure distribution, and W. N. MacDermott describes early operational experiences with an arc-heated blow-down hypersonic tunnel of about 0.03 sec. running time. H. H. Korst and W. Tripp predict the base pressure at the blunt trailing edge of a supersonic aerofoil at incidence; experiments confirm again the validity of the Korst model and suggest that an appreciable change in base pressure can arise from external pressure waves incident on the base region. A simplified model for the head-on interaction of shocks with rigid grids is developed by H. O. Barthel, and the characteristics systems of the equations governing interaction of a magnetic field with one-dimensional unsteady flow of a conducting gas are classified by S. I. Pai. R. W. Truitt proposes that the flow near the nose of a body in hypersonic flight be approximated by that generated when the body moves slowly towards a fixed concentric shell.

Some effects of chemical reactions on inviscid hydrodynamic stability are discussed by S. F. Shen, and Görtler draws attention to H. Oser's thesis on the vortex sheets arising in the low-frequency perturbations of rotating inviscid fluids, and to the analogous effect in fluids with stable density stratification. M. Lessen gives the compressible perturbation equations for flow with vortex-symmetry, and

A. de Neufville gives solutions for vortices decaying due to viscous shear diffusion. Some functions useful for the calculation of two-dimensional, incompressible laminar heat transfer from bodies with variable pressure and wall-temperature are tabulated by C. R. Guha and C. S. Yih, J. M. Robertson reviews data on turbulent flow in rough pipes, while E. R. G. Eckert and T. F. Irvine present some measurements on entrance length, transition and friction in non-circular pipes. D. W. Smith reports floating-element measurements of turbulent skin friction on a flat plate, V. A. Sandborn proposes a two-parameter family of mean profiles differing from Coles' (which reviewer prefers), and W. K. Mueller discusses why slow pulsations in pipe flow should reduce the average turbulent heat transfer slightly below that for steady mean flow.

P. Kaplan generalises Stokers work on the surface waves generated by pressure distributions to the case of uniformly moving oscillatory disturbances and shows that permanent advancing waves are formed also far ahead of the disturbance when the product of its frequency and speed is less than $g/4$. A. G. Strandhagen and G. R. Seikel calculate surface wave effects on the performance of a submerged hydrofoil of infinite span by the lifting line model, J. P. Breslin presents a simplified calculation of the suction exerted on a very slender body by a propeller at its stern, and F. E. Stelson and J. P. Murtha measure the virtual mass of some bodies moving in water. An ellipse-fitting approximation for calculations of two-dimensional impact of bodies on water by the "added-mass" model is proposed by G. Fabula. E. Silberman reports experiments confirming Rayleigh's prediction of the instability of gas jets in liquids, and W. H. Guilinger and E. Saibel discuss the programming of the approximate one-dimensional equations of adiabatic slider bearings. A little philosophical accompaniment is provided by M. M. Munk, M. Z. v. Krzywoblocki and C. G. Whittenbury.

R. E. Meyer

Differential equations: geometric theory. By S. Lefschetz. Interscience Publishers, Inc., New York and London, 1957. x + 364 pp. $9.50.

This book is strongly to be recommended to anyone entering the domain of differential equations, either as a pure or applied mathematician. The author has presented in a very readable and flowing style the fundamentals of the theory of the existence and stability of equilibrium and periodic solutions of nonlinear ordinary differential equations.

The title does not do the contents justice. In addition to presenting the geometric theory, in the sense of trajectories, critical points, and periodic solutions, the author gives a very lucid and detailed discussion of the analytic stability theory of Poincaré-Lyapunov. Starting with the ideas of these authors, he continues to the more recent work of Perron, Bellman, Levinson, Persidiski, Malkin, Massera and Antosiewicz.

Remarkably, in addition to a quite detailed discussion of systems of n-th order, and then of the second order, he presents a survey of various approximate techniques devised by Van der Pol and extensively developed by Krylov-Bogoliubov, and of the more recent stroboscopic method of Minorsky. These methods are illustrated by examples.

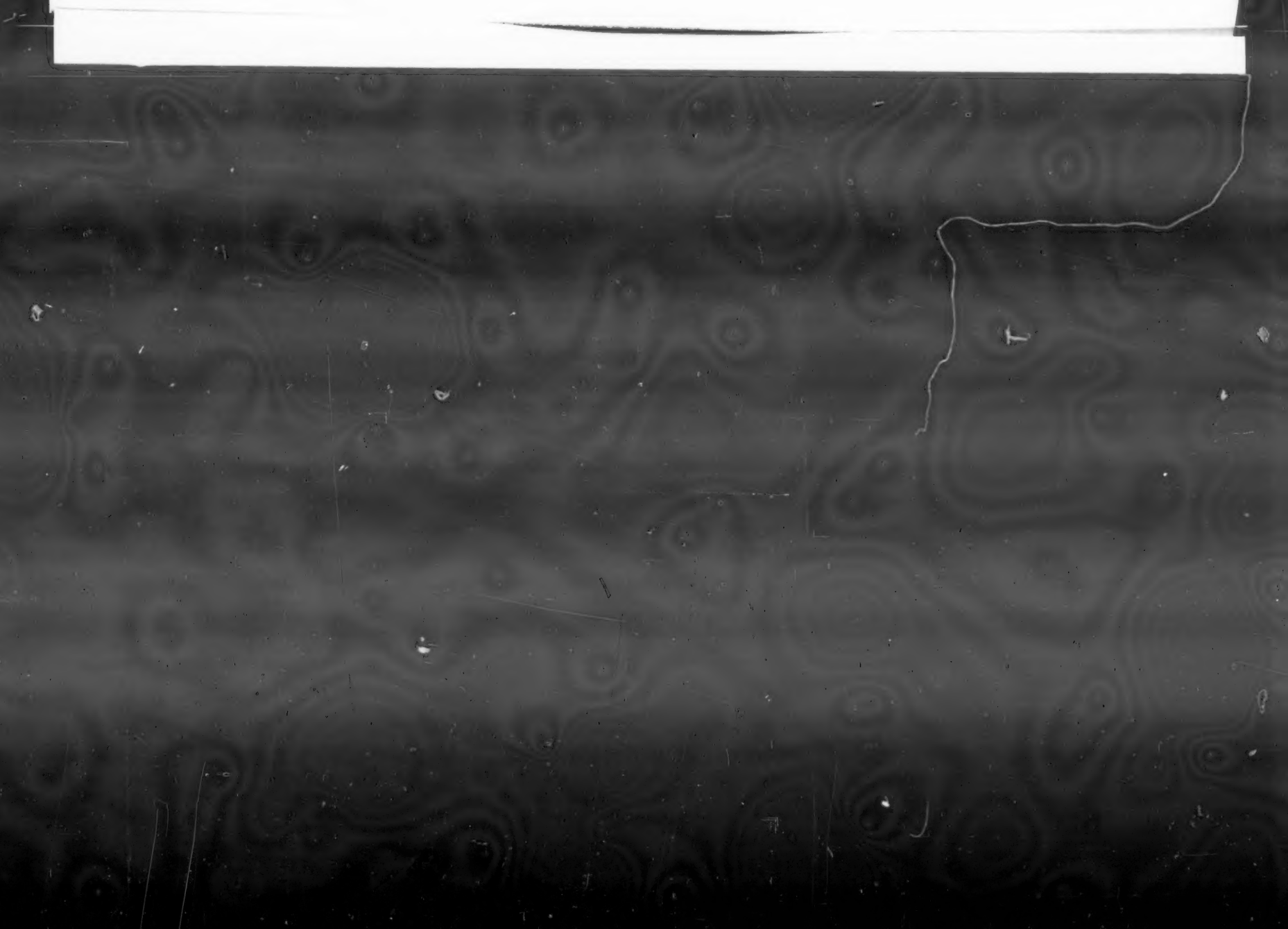
Richard Bellman

Integraltefeln zur Quantenchemie. By H. Preuss. Volume II. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1957. 143 pp. $8.60.

This volume consists of two parts, in the first of which the integrals and the method of calculation are described along with the details of the one electron wave function used in the calculation of molecular energies. The second part of this volume is concerned with the numerical tabulation of the results of the calculation of the interaction integrals for the K and L Shells as a function of the screening parameters used in the normalized Slater functions.

These tables constitute the second volume of such integrals for quantummechanical calculations. This work is part of a program of the Max Planck Institute for Physics in Göttingen to prepare such tables for the more important types of calculation.

Rohn Truell



CONTENTS

P. O. Bent: Some applications of differential geometry to the investigation of physical problems 321
E. L. Stiefel, J. S. Sumner, and S. R. Zoom: Multiple Fourier analysis in rectifier problems 345
R. P. Kanwal: On curved shock waves in the solar corona and gas flows 361
Lu Ting: Some integral properties of solutions of the wave equation with non-planar boundaries 373
W. Wasow and G. Hagedorn: Runge-Kutta formulas 385
[illegible] 394

Notices

[illegible] 405
M. Schiffer: A general variational principle for the solution of two-dimensional problems 413
C. B. Dreyfus: Formulas for the multiple integral of the Hill equation 421
A. Weinstein: Stability and vibration problems in hydrodynamics 425
R. C. Gunness: A boundary integral equation method for the solution of certain boundary value problems 426
E. Pinney: On a generalization of a theorem of Lagrange 461
I. Cohen: Examples of non-uniqueness in the generalized Cauchy problem 462
G. F. Carrier, and C. E. Pearson: On the solution of the problem of vortex shedding behind a circular cylinder 469
P. Kirrmann: Über die Integration der Differentialgleichungen mit [illegible] 490

Book Reviews 504, 509, 511

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